## **Exercise 4**

## Task 1

Consider the following algorithm, which tests probabilistically if AB = C for given matrices  $A, B, C \in \mathbb{Z}^{n \times n}$ :

- 1. Choose a vector  $v \in \{0,1\}^{n \times 1}$  randomly and uniformly distributed.
- 2. Compute w = A(Bv) Cv.
- 3. If w = 0 return "yes", otherwise "no".

Prove that in the case  $AB \neq C$  the algorithm returns "yes" with a probability of at most  $\frac{1}{2}$ .

## Task 2

Let G = (V, E) be an undirected graph with

$$V = \{1, 2, 3, 4, 5, 6\}, E = \{\{1, 3\}, \{1, 6\}, \{2, 3\}, \{2, 5\}, \{3, 5\}, \{4, 6\}, \{5, 6\}\}.$$

- (a) Compute the Tutte matrix  $T_G$  of G.
- (b) Compute the polynomial  $\det(T_G)$ .
- (c) Does G have a perfect matching? If yes, name all perfect matchings of G. If no, justify your answer.

## Task 3

Let G = (V, E) be an undirected graph with  $V = \{1, \ldots, n\}$ . Let  $T^G = (T_{u,v})_{1 \le u,v \le n}$  be the matrix defined by

$$T_{u,v} = \begin{cases} x_{u,v} & \text{if } \{u,v\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

- (a) Let G be a bipartite graph. This means, there are disjoint subsets  $U, W \subset V$  such that  $V = U \cup W$  and  $\{u, w\} \in E$  only if  $u \in U$  and  $w \in W$  (or  $w \in U$  and  $u \in W$ ). Show that G has a perfect matching if and only if  $\det(T^G) \neq 0$ .
- (b) Does (a) also hold, if G is not bipartite?