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Exercise 2

Task 1. If $|x\rangle$ is a unit vector and $\{|x_1\rangle, ..., |x_d\rangle\}$ an orthonormal base then $|x\rangle = \Sigma_{i=1}^d \alpha_i |x_i\rangle$ for unique $\alpha_1, ..., \alpha_d \in \mathbb{C}$ with $\Sigma_{i=1}^d |\alpha_i|^2 = 1$.

Task 2. Let f be a linear mapping and let $\{|x_1\rangle, .., |x_d\rangle\}$ and $\{|y_1\rangle, ... |y_d\rangle\}$ be two bases of $\mathbb{C}^d$. Let A (resp., B) be the matrix for f in the basis $\{|x_1\rangle, .., |x_d\rangle\}$ (resp., $\{|y_1\rangle, ... |y_d\rangle\}$).
Then, there is an invertible matrix $C \in \mathbb{C}^{d\times d}$ such that $B = C^{-1}AC$.
Find the matrix C explicitly.

Task 3. The trace of a matrix $\mathrm{tr}(A)$ is defined as the sum of the diagonal entries:

$$\mathrm{tr}(A) = \Sigma_{i=1}^d A_{i,i},$$

then, prove the following important properties:

- $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$
- $\mathrm{tr}(\alpha A) = \alpha \cdot \mathrm{tr}(A)$
- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

Task 4. If Π is any projector ($\Pi^2 = \Pi$) find a subspace S with $\Pi = \Pi_S$.

Task 5. Calculate the eigenvalues of the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Task 6. Prove that for every matrix $A \in \mathbb{C}^{d\times d}$ the matrix $A^\dagger A$ is positive semi-definite.

Task 7. Let A and B be unitary Hermitian positive definite projectors, then $A \otimes B$ is an unitary Hermitian positive definite projector.