Logical Aspects of Cayley-Graphs: The Group Case

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Abstract. We prove that a finitely generated group is context-free whenever its Cayley-graph has a decidable monadic second-order theory. Hence, by the seminal work of Muller and Schupp, our result gives a logical characterization of context-free groups and also proves a conjecture of Schupp. To derive this result, we investigate general graphs and show that a graph of bounded degree with a high degree of symmetry is context-free whenever its monadic second-order theory is decidable. Further, it is shown that the word problem of a finitely generated group is decidable if and only if the first-order theory of its Cayley-graph is decidable.

1 Introduction

Cayley-graphs of finitely generated groups are a fundamental concept in group theory. They were introduced by Cayley [12] for finite groups and Dehn [19] for infinite groups. Given a finite set Γ of generators of a group $\mathcal{G}$, the Cayley-graph of $\mathcal{G}$ with respect to Γ is a directed graph with node set $\mathcal{G}$, which contains an a-labeled edge (where $a \in \Gamma$) from $x \in \mathcal{G}$ to $y \in \mathcal{G}$ if and only if $y = xa$ in $\mathcal{G}$. Many deep results in group theory use Cayley-graphs in an essential way, see e.g. [20, 38, 57, 60]. Moreover, Cayley-graphs turned out to be a link to several other fields in mathematics and theoretical computer science, e.g., automata theory, topology, and graph theory.

In this paper we will investigate the logical aspects of Cayley-graphs and relate these aspects to the word problem of groups. The word problem of a group may be viewed as a formal language containing all words over the generators that represent the identity of the group. It turned out that the grammatical properties of the word problem, in particular its level in the Chomsky hierarchy, are related to the algebraic properties of the group, see e.g. [2, 7, 33, 34, 37, 44, 45, 59] for important results in this direction. The seminal work of Muller and Schupp [44, 45] investigates relations between the word problem and properties of the Cayley-graph: the word problem is a context-free language if and only if the Cayley-graph is the transition graph of a pushdown automaton [45] - this gives ample reason for calling these groups *context-free*. Muller and Schupp also presented a graph theoretical characterization of the transition graphs of pushdown automata (which they called context-free graphs) and proved that every context-free graph has a decidable monadic second-order theory [45]. Hence, the monadic second-order theory of the Cayley-graph of a context-free group is decidable. Here, we prove the converse of this statement: the context-free groups are the only groups whose Cayley-graph has a decidable monadic second-order theory (Corollary 4.1). This result proves a conjecture of Schupp from [52].

The fact that the monadic second-order theory of any context-free graph is decidable has spurred further attempts to extend this decidability result. Courcelle [14] proved this for equational graphs as well as for the class of all graphs of tree-width uniformly bounded by some constant. The former was extended by Caucal to prefix-recognizable graphs [9]. Similarly, in the course of proving our above-mentioned result for Cayley-graphs, we will study general graphs with decidable monadic second-order theories in Section 3. Our main tool for the investigation of these graphs are strong tree decompositions, and we show several combinatorial properties concerning these decompositions (see Section 3.2). Using these properties in combination with results of Seese, Courcelle, Muller, and Schupp, we are able to prove the main result of Section 3 (Theorem 3.10): a connected graph of bounded degree, whose automorphism group has only finitely many orbits, has a decidable monadic second-order theory if and only if it is context-free. The above mentioned characterization of those groups whose Cayley-graphs have decidable monadic second-order theories is an immediate consequence of Theorem 3.10.

In Section 4.3 we will prove a similar result for first-order logic: The first-order theory of the Cayley-graph of a group is decidable if and only if the word problem of the group is decidable (Theorem 4.9). For the proof of this result we apply a technique developed by Ferrante and Rackoff [30] which is based on Ehrenfeucht-Fraïssé games. We introduce this method in a slight variant in Section 4.2. In addition to the statement of Theorem 4.9, the method of Ferrante and Rackoff also provides an upper bound for the complexity of the first-order theory of the Cayley-graph in terms of the complexity of the word problem (Theorem 4.8). Finally, we prove that the word problem is recursively enumerable if and only if the positive first-order theory of the Cayley-graph, which contains all sentences from the full first-order theory that do not use negations, is recursively enumerable (Theorem 4.12).

Our results on first-order theories of Cayley-graphs should be also compared with the classical results about first-order theories of groups: the first-order theory of a group $\mathcal{G}$ contains all true first-order statements about $\mathcal{G}$ that are built over the signature containing the group operation and all group elements as constants. Thus, the first-order theory of the Cayley-graph of $\mathcal{G}$ can be seen as a fragment of the whole first-order theory of $\mathcal{G}$ in the sense that only equations of the form $xa = y$, with x and y variables and $a \in \mathcal{G}$ are allowed. In this context we should mention the classical results of Makanin, stating the decidability of the existential first-order theory and positive first-order theory of a free group [41, 42], which were extended in [21–23, 49] to larger classes of groups.

In a forthcoming paper, we investigate the logical aspects of Cayley-graphs of monoids. There, we show in particular that the class of monoids whose Cayley-graph has a decidable monadic second-order theory is closed under free products, and that the class of monoids whose Cayley-graph has a decidable first-order theory is closed under graph products (for groups these closure properties are simple corollaries of Corollary 4.1 and Theorem 4.9). A complete characterization in the style of Corollary 4.1 and Theorem 4.9 seems beyond our reach.

Some of the results of this paper can be found in the extended abstract [36].

2 Preliminaries

This section collects concepts from mathematical logic and combinatorial group theory that will play a central role in this paper. A broad introduction into mathematical logic can be found in [35], for more details on monadic second-order logic see [32]. For a further investigation of combinatorial group theory, the reader is refered to [39, 40].

Let Γ be a finite alphabet. The empty word over Γ is denoted ε. The length of a word $s \in \Gamma^*$ is denoted by $|s|$.

Relational structures and logic The notion of a structure (or model) is defined as usual in logic, see e.g. [35]. Here we only consider *relational structures*. Sometimes, we will also use constants, but a constant c can be always replaced by the unary relation $\{c\}$. Let us fix a relational structure $\mathcal{A} = (A, (R_i)_{i\in J})$, where $R_i \subseteq A^{n_i}$ for $i \in J$. The *signature of* $\mathcal{A}$ contains the equality symbol $=$, and for each $i \in J$ it contains a relation symbol of arity n_i that we denote without risk of confusion by R_i as well. For $B \subseteq A$ we define the restriction $\mathcal{A}{\restriction}B = (B, (R_i \cap B^{n_i})_{i\in J})$, it is a structure over the same signature as $\mathcal{A}$. Let $\mathcal{A}\backslash B = \mathcal{A}{\restriction}(A\backslash B)$. Given further relations R_j ($j \in K$, $J \cap K = \emptyset$) we also write $(\mathcal{A}, (R_i)_{i\in K})$ for the structure $(A, (R_i)_{i\in J\cup K})$. With $\mathrm{Aut}(\mathcal{A})$ we denote the *automorphism group* of $\mathcal{A}$. On the universe A we define the equivalence relation $\sim$ by $a \sim b$ if there exists $f \in \mathrm{Aut}(\mathcal{A})$ with $f(a) = b$. The equivalence classes of $\sim$ are called the *orbits* of $\mathrm{Aut}(\mathcal{A})$ on $\mathcal{A}$.

Next, let us introduce *monadic second-order logic (MSO-logic)*. Let $\mathbb{V}_1$ be a countably infinite set of *first-order variables* which range over elements of the universe A. First-order variables are denoted x, y, z, x', etc. Let $\mathbb{V}_2$ be a countably infinite set of *second-order variables* which range over subsets of A. Variables from $\mathbb{V}_2$ are denoted X, Y, Z, X', etc. *MSO-formulas* over the signature of $\mathcal{A}$ are constructed from the atomic formulas $R_i(x_1, \dots, x_{n_i})$, $x = y$, and $x \in X$ (where $i \in J$, $x_1, \dots, x_{n_i}, x, y \in \mathbb{V}_1$, and $X \in \mathbb{V}_2$) using the Boolean connectives $\neg, \wedge$, and $\vee$, and quantifications over variables from $\mathbb{V}_1$ and $\mathbb{V}_2$. The notion of a free occurrence of a variable is defined as usual. A formula without free occurrences of variables is called an *MSO-sentence*. If $\varphi(x_1,\dots,x_n,X_1,\dots,X_m)$ is an MSO-formula such that at most the first-order variables among $x_1,\dots,x_n$ and the second-order variables among $X_1,\dots,X_m$ occur freely in φ, and $a_1,\dots,a_n \in A$, $A_1,\dots,A_m \subseteq A$, then $\mathcal{A} \models \varphi(a_1,\dots,a_n,A_1,\dots,A_m)$ means that φ evaluates to true in $\mathcal{A}$ if the free variable x_i (resp. X_j) evaluates to a_i (resp. A_j). The *MSO-theory* of $\mathcal{A}$, denoted by $\mathrm{MSOTh}(\mathcal{A})$, is the set of all MSO-sentences φ such that $\mathcal{A} \models \varphi$.

A *first-order formula* over the signature of $\mathcal{A}$ is an MSO-formula that does not contain any occurrences of second-order variables. In particular, first-order formulas do not contain atomic subformulas of the form $x \in X$. The *first-order theory* $\mathrm{FOTh}(\mathcal{A})$ of $\mathcal{A}$ is the set of all first-order sentences φ such that $\mathcal{A} \models \varphi$. The *positive first-order theory* $\mathrm{posFOTh}(\mathcal{A})$ of $\mathcal{A}$ is the set of all sentences in $\mathrm{FOTh}(\mathcal{A})$ that do not contain the negation symbol $\neg$, i.e., only the Boolean connectives $\wedge$ and $\vee$ are allowed. The *existential first-order theory* $\exists\mathrm{FOTh}(\mathcal{A})$ of $\mathcal{A}$ is the set of all sentences in $\mathrm{FOTh}(\mathcal{A})$ of the form $\exists x_1 \cdots \exists x_n\ \varphi(x_1,\dots,x_n)$, where $\varphi(x_1,\dots,x_n)$ is a Boolean combination of atomic formulas. The *quantifier-depth* of a first-order formula φ is the maximal number of nested quantifiers in φ.

Word problems for groups Let $\mathcal{G}$ be a finitely generated group with identity 1 and let Γ be a finite (monoid) generating set for $\mathcal{G}$, i.e., there exists a surjective monoid homomorphism $h : \Gamma^* \to \mathcal{G}$. We will always assume that Γ is closed under taking inverses, i.e., $a \in \Gamma$ implies that also $a^{-1} \in \Gamma$.[3] The *word problem* for $\mathcal{G}$ with respect to Γ is the set $W(\mathcal{G},\Gamma) = \{w \in \Gamma^* \mid h(w) = 1\}$. The following facts are well-known, see e.g. [33] for a proof of the second statement.

Theorem 2.1. *Let $\mathcal{G}$ be a finitely generated group and let Γ_1 and Γ_2 be two finite generating sets for $\mathcal{G}$. Then the following holds:*

- *$W(\mathcal{G},\Gamma_1)$ is logspace reducible to $W(\mathcal{G},\Gamma_2)$.*[4]
- *If $\mathcal{C}$ is some class of languages that is closed under inverse morphisms, then $W(\mathcal{G},\Gamma_1) \in \mathcal{C}$ if and only if $W(\mathcal{G},\Gamma_2) \in \mathcal{C}$.*

By the first statement, the computational complexity of the word problem does not depend on the underlying set of generators. By the second statement, it is independent of the underlying set of generators whether the word problem is regular (resp. context-free, context-sensitive, decidable, recursively enumerable). Therefore, it makes sense to say that the word problem for $\mathcal{G}$ is regular (resp. context-free, context-sensitive, decidable, recursively enumerable). The following theorem presents algebraic characterizations:

[3] Hence, by choosing a subset Σ of Γ such that $\Gamma = \Sigma \cup \{a^{-1} \mid a \in \Sigma\}$, we obtain a group generating set Σ for $\mathcal{G}$. Moreover, we can factorize the monoid homomorphism $h : \Gamma^* \to \mathcal{G}$ as $h = h_1 \circ h_2$, where $h_1 : \Gamma^* \to F(\Sigma)$ is the canonical homomorphism from the free monoid Γ^* to the *free group* $F(\Sigma)$ generated by Σ and $h_2 : F(\Sigma) \to \mathcal{G}$ is a group homomorphism.

[4] See e.g. [46] for the notion of logspace reducibility.

Theorem 2.2. *Let $\mathcal{G}$ be a finitely generated group. The following holds:*

- *$\mathcal{G}$ has a regular word problem if and only if $\mathcal{G}$ is finite [2].*
- *$\mathcal{G}$ has a context-free word problem if and only if $\mathcal{G}$ is* virtually-free*, i.e., has a free subgroup of finite index [27, 44].*
- *$\mathcal{G}$ has a decidable word problem if and only if $\mathcal{G}$ can be embedded in a simple subgroup of a finitely presented group [7].*
- *$\mathcal{G}$ has a recursively enumerable word problem if and only if $\mathcal{G}$ can be embedded in a finitely presented group [34].*

Groups with a context-free word problem are also called *context-free groups*.

Cayley-graphs of groups Cayley-graphs play an important role in combinatorial group theory [39, 40], see also the surveys of Babai [3] and Schupp [51]. Let $\mathcal{G} = (G, \circ, 1)$ be a finitely generated group and Γ be a finite generating set of $\mathcal{G}$. The *Cayley-graph* of $\mathcal{G}$ with respect to Γ is the following relational structure:

$$\mathcal{C}(\mathcal{G}, \Gamma) = (G, (\{(u, v) \mid u \circ a = v\})_{a \in \Gamma})$$

It can be viewed as a directed graph where every edge has a label from Γ and $\{(u, v) \mid u \circ a = v\}$ is the set of a-labeled edges. We express the fact that there exists an a-labeled edge from x to y by writing $x \circ a = y$ or briefly $xa = y$. Since Γ generates $\mathcal{G}$, $\mathcal{C}(\mathcal{G}, \Gamma)$ is a connected graph. Moreover, there exists a reversed a^{-1}-edge for every a-labeled edge of $\mathcal{C}(\mathcal{G}, \Gamma)$ ($a \in \Gamma$). One of the most important properties of Cayley-graphs is the fact that $\mathrm{Aut}(\mathcal{C}(\mathcal{G}, \Gamma))$ has only one orbit on $\mathcal{C}(\mathcal{G}, \Gamma)$: for every two nodes $u, v \in G$ there exists an automorphism of $\mathcal{C}(\mathcal{G}, \Gamma)$ that maps u to v.

Similarly to the word problem, the chosen generating set has no influence on the decidability (or complexity) of the first-order (resp. monadic second-order) theory of the Cayley-graph:

Proposition 2.3. *Let Γ_1 and Γ_2 be finite generating sets for the group $\mathcal{G}$. Then* $\mathrm{FOTh}(\mathcal{C}(\mathcal{G}, \Gamma_1))$ *is logspace reducible to* $\mathrm{FOTh}(\mathcal{C}(\mathcal{G}, \Gamma_2))$ *and the same holds for the monadic second-order theories.*

Proof. The arguments for first-order logic and MSO-logic, respectively, are the same. Thus, we only consider the first-order case. Given a first-order sentence ϕ_1 over the signature of $\mathcal{C}(\mathcal{G}, \Gamma_1)$ we construct a first-order sentence ϕ_2 over the signature of $\mathcal{C}(\mathcal{G}, \Gamma_2)$ such that $\mathcal{C}(\mathcal{G}, \Gamma_1) \models \phi_1$ if and only if $\mathcal{C}(\mathcal{G}, \Gamma_2) \models \phi_2$ as follows: Let $a \in \Gamma_1$. Then there exists a word $b_0 b_1 \cdots b_{n-1}$ with $b_i \in \Gamma_2$ such that a and $b_0 b_1 \cdots b_{n-1}$ represent the same group element of $\mathcal{G}$. Then we replace every occurrence of the formula $xa = y$ by

$$\exists z_0 \cdots \exists z_n \left\{ \bigwedge_{0 \le i < n} z_i b_i = z_{i+1} \ \wedge\ z_0 = x \ \wedge\ z_n = y \right\},$$

this can be done in logspace. By doing this replacement for every $a \in \Gamma_1$, we obtain ϕ_2. □

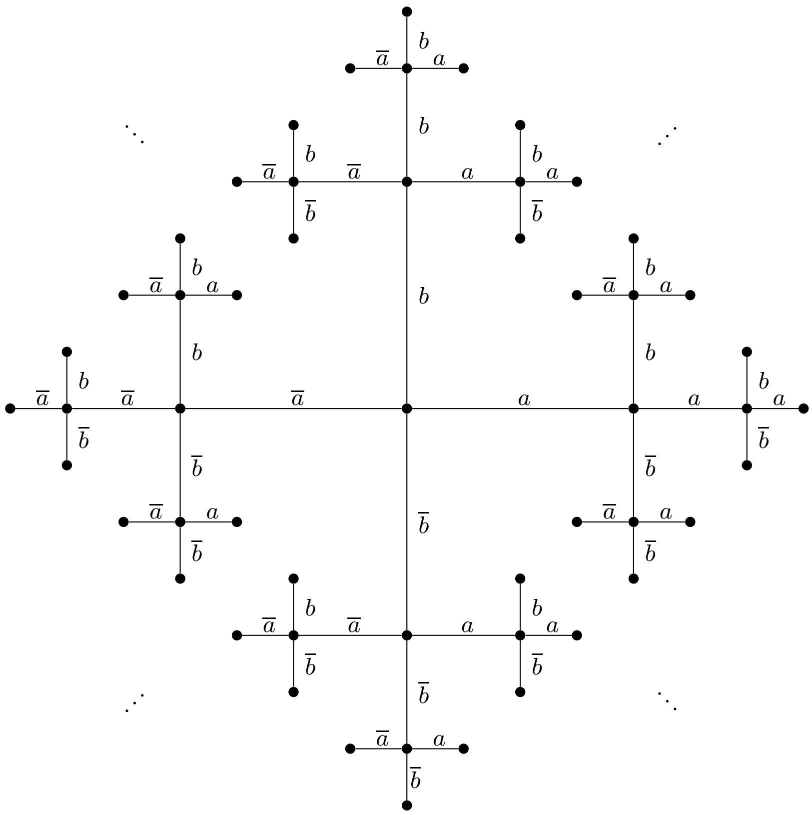


Fig. 1. The Cayley-graph of F_2

Whenever the specific generating set Γ will be of no importance, we will briefly write $\mathcal{C}(\mathcal{G})$ instead of $\mathcal{C}(\mathcal{G}, \Gamma)$.

Figure 1 and 2 depict some typical Cayley-graphs. There, two edges which are reversed to each other are represented as a single undirected edge with the label of the edge that points away from the origin, i.e., from the node representing the identity 1. Figure 1 shows the Cayley-graph of F_2, the free group of rank 2, with respect to the standard generating set $\{a, a^{-1}, b, b^{-1}\}$. This Cayley-graph is a complete tree of degree 4. Hence, by Rabin's tree theorem [48], $\mathrm{MSOTh}(\mathcal{C}(F_2))$ is decidable.

Figure 2 presents the Cayley-graph of $\mathbb{Z} \times \mathbb{Z}$ with respect to the generating set $\{a, a^{-1}, b, b^{-1}\}$ (with $ab = ba$). Since it is an infinite grid, its MSO-theory is undecidable, see e.g. [63]. But the first-order theory of this graph is still decidable.

The goal of the further investigations is to obtain complete characterizations of those groups $\mathcal{G}$ for which $\mathrm{MSOTh}(\mathcal{C}(\mathcal{G}))$ (resp. $\mathrm{FOTh}(\mathcal{C}(\mathcal{G}))$) is decidable.

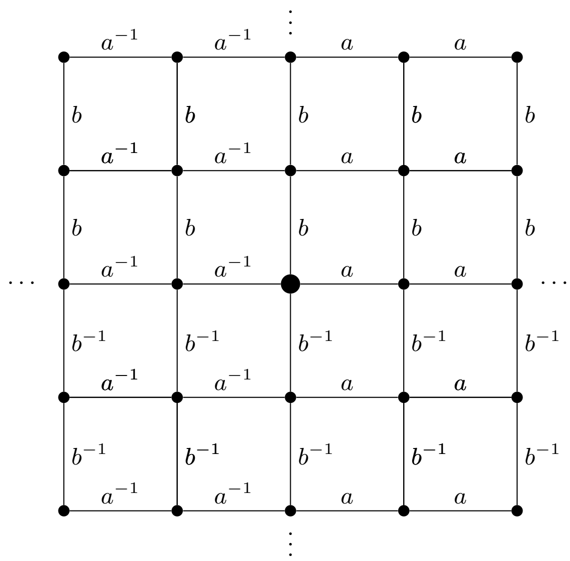


Fig. 2. The Cayley-graph of $\mathbb{Z} \times \mathbb{Z}$

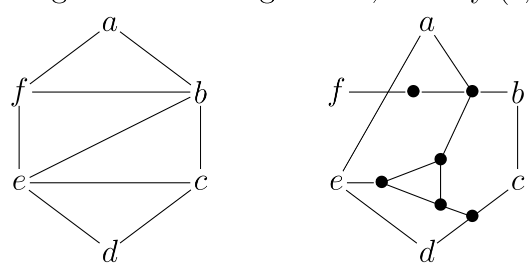
3 Graphs, tree-width, and MSO

In this section, we study general graphs and their MSO-theories. The motivation is two-fold: The results will lead to a complete characterization of those finitely generated groups $\mathcal{G}$ for which MSOTh($\mathcal{C}(\mathcal{G})$) is decidable. Further, it shows that Caucal's program [8, 10, 11] to identify classes of graphs all of whose members have a decidable MSO-theory can be completed for graphs of bounded degree with a high degree of symmetry.

3.1 Undirected graphs

An *undirected graph* is a relational structure $G = (V, E)$, where V is called the set of nodes and $E \subseteq V \times V$ is a symmetric and irreflexive edge relation (thus, undirected graphs do not have self loops). All notions that were defined for arbitrary relational structures in Section 2 will also be used for undirected graphs. We will also use the notation $V(G) = V$ and $E(G) = E$. A *path* of length $n \geq 0$ in G between $u \in V$ and $v \in V$ is a sequence $[v_0, v_1, \ldots, v_n]$ of nodes such that $v_0 = u$, $v_n = v$, and $(v_i, v_{i+1}) \in E$ for all $0 \leq i < n$; it is a *closed path* if $u = v$; it is a *simple path* if $v_i \neq v_j$ for $i \neq j$. We write $d_G(u, v)$ for the distance between the nodes $u, v \in V$, i.e., $d_G(u, v)$ is the minimal length of a path between u and v. If such a path does not exist, we write $d_G(u, v) = \infty$. The *r-sphere, centered at* $v \in V$, is $S_G(r, v) = \{u \in V \mid d_G(v, u) \leq r\}$. For a k-tuple $\widetilde{v} = (v_1, \ldots, v_k) \in V^k$ we define $S_G(r, \widetilde{v}) = \bigcup_{i=1}^{k} S_G(r, v_i)$. The graph G is *connected* if $d_G(u, v) < \infty$ for all $u, v \in V$. The graph G is *acyclic* if G does not contain a closed path $[v_1, v_2, \ldots, v_n, v_1]$ such that $n \geq 3$ and $[v_1, v_2, \ldots, v_n]$ is simple. A *forest* is an acyclic graph and a *tree* is a connected forest. Let $U \subseteq V$. The undirected graph $G{\restriction}U$ is called the *subgraph of G, induced by U*. The *diameter* $\mathrm{diam}_G(U)$ *of U* is the supremum in $\mathbb{N} \cup \{\infty\}$ of the set $\{d_G(u, v) \mid u, v \in U\}$. A *connected component* of G is an induced subgraph $G{\restriction}U$ such that $U = \{u \in V \mid d_G(v,u) < \infty\}$ for some node $v \in V$. The *degree* of a node $v \in V$ is the cardinality of the set $\{u \in V \mid (v,u) \in E\}$. The graph G is called *of bounded degree* if there exists some $d \in \mathbb{N}$ such that each node $v \in V$ has degree at most d. In this case we also say that G is *of bounded degree d*.

Let $\pi = [v_1, v_2, \dots, v_m, v_1]$ be a sequence of nodes $v_i \in V$ (which is not necessarily a path). With π we associate a closed convex polygon $\mathrm{Pol}(\pi)$ in the plane, whose boundary has m vertices $x_1, \dots, x_m$, which are labeled in clockwise order with $v_1, \dots, v_m$. For $M \geq 1$, an *M-triangulation* of π is a plane triangulation of $\mathrm{Pol}(\pi)$ with vertex set $\{x_1, x_2, \dots, x_m\}$ and additional edges of the form (x_i, x_j) for $d_G(v_i, v_j) \leq M$, only. We say that G can be *M-triangulated* if every *closed path* π of G can be M-triangulated [44, 66]. The example below shows a 3-triangulation of the sequence $[a,b,c,d,e,f,a]$ in the graph on the right. We have three additional edges in the triangulation, namely (b,e), (b,f), and (c,e).



A *tree decomposition* of $G = (V,E)$ is a pair (T,f), where T is a tree and $f : V(T) \to 2^V \setminus \{\emptyset\}$ is a function such that the following holds:

- $\bigcup_{w \in V(T)} f(w) = V$,
- for every $(u,v) \in E$ there exists $w \in V(T)$ such that $u, v \in f(w)$, and
- if $w_1, w_3 \in V(T)$ and w_2 lies on the unique simple path from w_1 to w_3 in the tree T, then $f(w_1) \cap f(w_3) \subseteq f(w_2)$.

The supremum in $\mathbb{N} \cup \{\infty\}$ of the cardinalities $|f(w)|$, $w \in V(T)$, is called the *width* of the tree decomposition. We say that G has *tree-width* $\leq b$ if there exists a tree decomposition of width $\leq b$. Finally G has *finite tree-width* if it has tree-width $\leq b$ for some $b \in \mathbb{N}$. The notion of tree-width was introduced in [50] and plays a central role in Robertson and Seymour's theory of graph minors, see e.g. [24] for an overview.

For the rest of Section 3, a strengthening of the notion of tree-width will be more important. The next section introduces this strengthening.

3.2 Strong tree decompositions

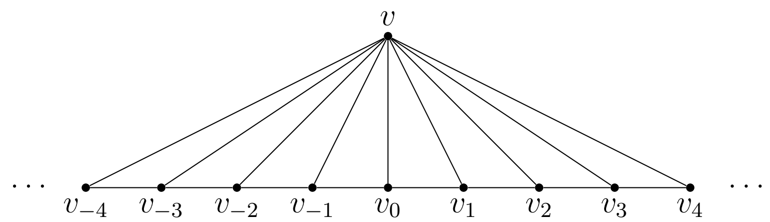
Let $G = (V,E)$ be an undirected graph and let P be a partition of V, i.e., $P \subseteq 2^V \setminus \{\emptyset\}$, $W_1 \cap W_2 = \emptyset$ for W_1 and W_2 distinct elements of P, and $\bigcup_{W \in P} W = V$. We define the *quotient graph* of G by P as the undirected graph

$$G/_P = (P,\ \{(W_1,W_2) \in P \times P \mid W_1 \neq W_2,\ (W_1 \times W_2) \cap E \neq \emptyset\}).$$

A *strong tree decomposition* of G is a partition P of V such that the quotient graph $G/_P$ is a forest. If G is connected and P is a strong tree decomposition of G, then $G/_P$ must also be connected, i.e., it is a tree. The *width* of a strong tree decomposition P is defined as the supremum in $\mathbb{N} \cup \{\infty\}$ of the cardinalities $|W|$ for $W \in P$. We say that G has *strong tree-width* $\leq b$ if there exists a strong tree decomposition of width $\leq b$. G has *finite strong tree-width* if it has strong tree-width $\leq b$ for some $b \in \mathbb{N}$. The notion of strong tree-width is taken from [53].

Any strong tree decomposition $\{W_i \mid i \in J\}$ gives rise to a tree decomposition formed by the sets $W_i \cup W_j$ whenever $W_i \times W_j$ contains some edge of the graph. Thus, a graph of finite strong tree-width has finite tree-width as well. On the other hand, the converse implication is false in general:

Example 3.1. Let G be the following graph of unbounded degree:



A tree decomposition of G of width 3 is (T, f) with

$$T = (\mathbb{Z}, \{(n, n+1), (n+1, n) \mid n \in \mathbb{Z}\})$$

(this is a tree) and $f(n) = \{v, v_n, v_{n+1}\}$. On the other hand, for every partition P of the set of nodes of G into at least three partition classes, $G/_P$ contains a triangle. Note also that $\mathrm{Aut}(G)$ has only two orbits on G.

Our first result of this section states that at least for graphs of bounded degree, finite tree-width implies finite strong tree-width. The second and in our context more important result is that under some conditions one can even find a strong tree decomposition of finite width such that all partition classes have a uniformly bounded diameter.

In [64] it is shown that an arbitrary graph G has tree-width $\leq b$ if and only if every finite subgraph of G has tree-width $\leq b$. By the next lemma, a corresponding statement for strong tree-width is true as well. We are grateful to Isolde Adler who provided us with the proof presented here. For countable graphs, an alternative proof can be given using König's Lemma.

Lemma 3.2. *Let $G = (V, E)$ be a graph. Then G has strong tree-width $\leq b$ if and only if every finite subgraph of G has strong tree-width $\leq b$.*

Proof. If G has strong tree-width $\leq b$, then every finite subgraph of G has strong tree-width $\leq b$. To prove the non-trivial implication, we use structures of the form $(V', E', \sim, (c_v)_{v \in V})$, where E' and $\sim$ are binary relations and c_v is a constant for every $v \in V$. Then we consider infinitely many first-order sentences expressing the following:

(1) the binary relation E' is symmetric and irreflexive
(2) the binary relation $\sim$ is an equivalence relation
(3) every equivalence class w.r.t. $\sim$ contains at most b elements
(4) the binary relation $\{(x,y) \in V' \times V' \mid \exists x' \exists y' : x \sim x' \wedge E'(x',y') \wedge y' \sim y\}$ (which is first-order definable) does not contain a cycle of length n (we write down such a sentence for every $n \in \mathbb{N}$)
(5) $c_v \neq c_w$ for $v, w \in V$ distinct
(6) $E'(c_v, c_w)$ for $v, w \in V$ if $(v,w) \in E$ and $\neg E'(c_v, c_w)$ otherwise

Let Φ be a finite subset of these sentences. Then this set mentions only finitely many of the constants c_v, i.e., there is a finite nonempty set $W \subseteq V$ such that at most the constants c_w for $w \in W$ appear in Φ. By our assumption on G, the subgraph of G induced by W has strong tree-width at most b. Let $\sim$ be an equivalence relation inducing such a strong tree decomposition and consider the structure $(W, E \cap (W \times W), \sim, (w)_{w \in W}, (u)_{v \in V \setminus W})$ where u is an arbitrary element of W. This structure satisfies all sentences from Φ (for $E' = E \cap (W \times W)$, $c_w = w$ for $w \in W$, and $c_v = u$ for $v \in V \setminus W$), i.e., Φ is satisfiable. Hence, compactness of first-order logic implies that the set of all first-order sentences above has a model $(V', E', \sim, (c_v)_{v \in V})$. In particular, (V', E') is an undirected graph. Let P be the partition of V' induced by the equivalence relation $\sim$. Then, since all sentences of the form (4) hold, the graph $(V', E')/_P$ does not contain any cycle, i.e., it is a forest. Hence, by the sentence (3), (V', E') has a strong tree decomposition of width at most b. Since all the sentences of the forms (5) and (6) hold, the graphs $(V', E')\lceil_{\{c_v \mid v \in V\}}$ and (V, E) are isomorphic, i.e., $G = (V, E)$ is an induced subgraph of (V', E'). Since the latter has strong tree width at most b, so does the former. $\square$

For related uses of compactness of first-order logic, see [1].

The following result on finite graphs was first stated in [6, Corollary 13]. It can be derived from a corresponding result for domino tree-width which was independently shown in [25]. Later, a simplified proof was given in [5].

Theorem 3.3 (cf. [5, 6]). *Let G be a finite graph of bounded degree $\leq d$ and tree-width $\leq b$. Then G has strong tree-width $\leq c(b, d)$, where c is a fixed function.*[5]

This result on finite graphs is the basis for our extension to infinite graphs as stated in the following theorem.

Theorem 3.4. *Let G be a graph of bounded degree. Then G has finite tree-width if and only if G has finite strong tree-width.*

Proof. If G has a strong tree decomposition P of width $\leq b$, then we can construct a tree decomposition of G of width $\leq 2b$ from all sets $W_1 \cup W_2$, where $W_1, W_2 \in P$ and $E(G) \cap (W_1 \times W_2) \neq \emptyset$.

[5] From the upper bound for domino tree-width given in [5], it follows that $c(b,d) = (9b+7)d(d+1)$ suffices.

Now assume that G has tree-width $\leq b$ and bounded degree d. Let $\mathcal{S}$ be the set of finite subgraphs of G. Then the tree-width of every graph in $\mathcal{S}$ is bounded by b [64]. Trivially, the degree of every graph in $\mathcal{S}$ is also bounded by d. From Theorem 3.3, we can infer that there is a constant c such that the strong tree-width of every graph in $\mathcal{S}$ is bounded by c. Hence, by Lemma 3.2, also G has strong tree-width $\leq c$. □

For our further considerations let us fix a connected graph $G = (V, E)$. If V is partitioned into sets V_1 and V_2, then the set of edges

$$C = E \cap [(V_1 \times V_2) \cup (V_2 \times V_1)]$$

is a *cut* of G. If $|C| \leq 2k$, then C is called a *k-cut* of G (we choose $2k$ here, since for undirected graphs, edges always come in pairs). The sets V_1 and V_2 are called the *sides* of the cut C. If both $G{\restriction}V_1$ and $G{\restriction}V_2$ are connected subgraphs of G, then C is called a *tight cut*. The importance of tight cuts in our context comes from the following result of Dunwoody [26, Paragraph 2.5]. Later a simplified proof was given in [65, Proposition 4.1].

Lemma 3.5 (cf. [26, 65]). *Let G be a connected graph and let $k \in \mathbb{N}$. Then every edge of G is contained in only finitely many tight k-cuts of G.*

Let P be a strong tree decomposition of the connected graph G and let $e = (W_1, W_2) \in E(G/_P)$ be an edge of $G/_P$. Since $\{e\}$ is a cut of the tree $G/_P$, we can define a cut

$$\text{cut}(e) = E \cap [(W_1 \times W_2) \cup (W_2 \times W_1)]$$

of G. We say that P is *tight* if $\text{cut}(e)$ is tight for all $e \in E(G/_P)$.

Lemma 3.6. *Let G be connected and of strong tree-width $\leq b$. Then there exists a tight strong tree decomposition of G of width $\leq b$.*

Proof. Let P be a strong tree decomposition of G of width $\leq b$. First we will refine P maximally to a strong tree decomposition Q, where Q is *finer than* P – written $Q \preceq P$ – if for any $W \in Q$, there is $W' \in P$ with $W \subseteq W'$. Then, we will show that Q is tight.

So let $(P_\alpha)_{\alpha<\kappa}$ be some decreasing chain (with respect to $\preceq$) of strong tree decompositions P_α with $P_0 = P$ where κ is some ordinal. If we order the sets in $\bigcup_{\alpha<\kappa} P_\alpha \subseteq 2^{V(G)}$ under set inclusion, we obtain a disjoint union of finite trees (one for each $W \in P$). Then the set Q of all minimal elements in $\bigcup_{\alpha<\kappa} P_\alpha$ (with respect to $\subseteq$) is a partition of $V(G)$. Assume that there is a cycle in $G/_Q$, involving the nodes $U_1, \ldots, U_m \in Q$. Then there is some $\alpha < \kappa$ with $U_1, \ldots, U_m \in P_\alpha$, contradicting our assumption that P_α is a strong tree decomposition. Thus, Q is a strong tree decomposition of G. We have shown that any decreasing chain of strong tree decompositions is bounded from below. By Zorn's Lemma, this implies the existence of a minimal (with respect to $\preceq$) strong tree decomposition Q in $\{P' \mid P' \preceq P\}$.

Suppose Q is not tight, i.e., there is an edge $e \in E(G/_Q)$ such that $\mathrm{cut}(e)$ is not tight. Let $U \subseteq V(G)$ be one of the sides of $\mathrm{cut}(e)$ such that $G \restriction U$ is not connected and let $U_j \subseteq U$ ($j \in J$) be the node sets of the connected components of $G \restriction U$. Let

$$P' = \{W \cap U_j \mid W \in Q,\ j \in J,\ W \cap U_j \neq \emptyset\} \cup \{W \in Q \mid W \subseteq V(G) \setminus U\}.$$

Then P' is a strong tree decomposition of G that is finer than Q, because if $e = (W_1, W_2)$, then at least the W_i with $W_i \subseteq U$ will be refined into more than one partition class of P'. We have obtained a contradiction. □

The next theorem is the main result of this section. Recall the notion of an orbit, which was defined in Section 2 for arbitrary relational structures.

Theorem 3.7. *Let G be a connected graph of bounded degree and of finite tree-width such that $\mathrm{Aut}(G)$ has only finitely many orbits on G. Then there exists a strong tree decomposition P of G of finite width and a constant c such that for all $W \in P$, $\mathrm{diam}_G(W) \leq c$.*

Proof. By Theorem 3.4, $G = (V, E)$ has strong tree-width $\leq b$ for some constant b. Thus, by Lemma 3.6 there exists a tight strong tree decomposition P of G of width $\leq b$. Hence, for all $e \in E(G/_P)$, $\mathrm{cut}(e)$ is a tight b^2-cut. In the following, for a cut $C \subseteq E$ let $V(C)$ denote the set of all $u \in V$ such that $(u, v) \in C$ for some $v \in V$. Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be the orbits of $\mathrm{Aut}(G)$ on G. For every $1 \leq i \leq n$ choose a node $v_i \in \mathcal{O}_i$. Let $\mathcal{C}$ be the set of all tight b^2-cuts C such that $V(C) \cap \{v_1, \ldots, v_n\} \neq \emptyset$. Then $\mathcal{C}$ is finite since every node v_i has only finitely many adjacent edges and, by Lemma 3.5, each of these edges is contained in only finitely many tight b^2-cuts. Since G is connected, we can therefore define $d = \max\{\mathrm{diam}_G(V(C)) \mid C \in \mathcal{C}\} \in \mathbb{N}$.

Let $e \in E(G/_P)$ and $v \in V(\mathrm{cut}(e))$. Then v can be mapped by some $f \in \mathrm{Aut}(G)$ to some v_i. But then the automorphism f maps $\mathrm{cut}(e)$ to some cut from $\mathcal{C}$. Thus, $\mathrm{diam}_G(V(\mathrm{cut}(e))) \leq d$.

Now let $W \in P$ and let $e_1, \ldots, e_m \in E(G/_P)$ be all those edges that are adjacent with W in $G/_P$. Let $V_i = V(\mathrm{cut}(e_i)) \cap W$. Thus, also $\mathrm{diam}_G(V_i) \leq d$ for all $1 \leq i \leq m$. Choose $u, v \in W$ with $u \neq v$. We will show that $d_G(u, v) \leq bd - 1$ which proves the theorem. Since G is connected, we can choose a simple path π in G between u and v of minimal length. Since $G/_P$ is a tree, we can split the path π into subpaths $\pi_1, \nu_1, \pi_2, \nu_2, \ldots, \pi_\ell, \nu_\ell, \pi_{\ell+1}$ ($\ell \geq 0$) such that

- π_1 starts in u, $\pi_{\ell+1}$ ends in v, and the final node of π_i (resp. ν_i) is the initial node of ν_i (resp. π_{i+1}),
- for all i, π_i is completely contained in W, and

– for all i, there exists j such that both the initial and final node of ν_i belong to V_j and are different; thus the length of ν_i is bounded by $d \geq 1$.

Since π is simple, the sum of the lengths of all paths π_i is at most $|W| - 1 - \ell \leq b - 1 - \ell$, and moreover $\ell \leq |W| \leq b$. It follows that the length of π is bounded by $b - 1 - \ell + \ell \cdot d = b - 1 + \ell(d-1) \leq b - 1 + b(d-1) = bd - 1$. □

3.3 Labeled directed graphs

Let Γ be some finite alphabet of labels. A *Γ-labeled directed graph* is a relational structure $G = (V, (E_a)_{a\in\Gamma})$ where V is the set of nodes and $E_a \subseteq V \times V$ is the set of a-labeled directed edges. Note that self-loops are allowed in directed graphs. The Cayley-graph $\mathcal{C}(\mathcal{G}, \Gamma)$ of a group $\mathcal{G}$ with respect to the finite generating set Γ is an example of a Γ-labeled directed graph. Let us fix $G = (V, (E_a)_{a\in\Gamma})$ for the further discussion. We associate with G the unlabeled undirected graph

$$\mathrm{ud}(G) = (V, \bigcup_{a\in\Gamma}\{(u,v) \mid u \neq v,\ (u,v) \in E_a \text{ or } (v,u) \in E_a\}).$$

We say that G is connected (resp. of bounded degree) if $\mathrm{ud}(G)$ is connected (resp. of bounded degree). Note that, for $U \subseteq V$, the structures $G{\restriction}U$ and $G \setminus U$ are also Γ-labeled directed graphs. A *connected component* of G is a subgraph $G{\restriction}U$, where $\mathrm{ud}(G){\restriction}U$ is a connected component of $\mathrm{ud}(G)$. For a node $v \in V$ we call the structure (G, v) a *rooted graph*.

Assume now that G is connected and of bounded degree (and thus countable), and let $v_0 \in V$ be a distinguished node. Let $v \in V \setminus \{v_0\}$ and $r = d_{\mathrm{ud}(G)}(v_0, v) - 1$. The unique connected component of $G \setminus S_{\mathrm{ud}(G)}(r, v_0)$ that contains the node $v \in V$ is denoted by $G(v)$. Furthermore, let

$$\Delta(v) = \{u \in V \mid u \text{ belongs to } G(v),\ d_{\mathrm{ud}(G)}(v_0, u) = r + 1\}.$$

Two subgraphs $G(u)$ and $G(v)$ are called *end-isomorphic* if there exists a (label-preserving graph-) isomorphism from $G(u)$ to $G(v)$ which bijectively maps $\Delta(u)$ to $\Delta(v)$. We say that the rooted graph (G, v_0) is *context-free* if there exist only finitely many $G(v)$ ($v \in V$) that are pairwise not end-isomorphic. This notion was introduced in [45], where it was shown that if (G, v_0) is context-free, then (G, u) is context-free for every $u \in V$. Hence, in this case we can say that the graph G is context-free. By [45] the context-free graphs are exactly the transition graphs of pushdown automata. Moreover, by a reduction to Rabin's tree theorem [48], Muller and Schupp have shown that every context-free graph has a decidable MSO-theory.

3.4 Monadic second-order logic over graphs

We begin this section with several known results related to MSO-logic over graphs, see [16] for a more comprehensive exposition.

Let us fix an undirected (unlabeled) graph $H = (V, E)$. Note that MSO-logic as introduced in Section 2 only allows second-order quantifications over subsets of V. In order to allow also quantifications over sets of edges, we introduce, following [15], an extended representation of graphs. More precisely, we define the relational structure

$$H^{(e)} = (V \cup E, \mathrm{inc}),$$

where $\mathrm{inc} = \{(e, v) \in E \times V \mid \exists u \in V : e \in \{(u, v), (v, u)\}\}$. We have introduced this extended representation of graphs because of the following important result of Seese, see also [16, Theorem 5.8.10].

Theorem 3.8 (cf. [54]). *Let H be an undirected graph. If $\mathrm{MSOTh}(H^{(e)})$ is decidable, then H has finite tree-width.*

This theorem holds even for classes of graphs. The converse of Seese's Theorem is not true: Using an undecidable subset of $\mathbb{N}$ it is easy to construct a tree with an undecidable first-order theory. On the other hand, Courcelle has shown that for every $b \in \mathbb{N}$ the class of *all* graphs of tree-width at most b has a decidable monadic second-order theory [14].

Note that if $\mathrm{MSOTh}(H^{(e)})$ is decidable, then also $\mathrm{MSOTh}(H)$ is decidable. For the reverse implication, restrictions on the graph H are necessary, e.g., for the complete graph on countably many nodes $H = K_{\aleph_0}$, $\mathrm{MSOTh}(H)$ is decidable but $\mathrm{MSOTh}(H^{(e)})$ is not. Courcelle has shown in [15] that for an undirected graph H of bounded degree, if $\mathrm{MSOTh}(H)$ is decidable, then also $\mathrm{MSOTh}(H^{(e)})$ is decidable.[6] Since for a Γ-labeled directed graph G, the decidability of $\mathrm{MSOTh}(G)$ implies the decidability of $\mathrm{MSOTh}(\mathrm{ud}(G))$, Theorem 3.8 implies the following result:

Theorem 3.9 (cf. [15, 54]). *Let G be a Γ-labeled directed graph of bounded degree. If $\mathrm{MSOTh}(G)$ is decidable, then $\mathrm{ud}(G)$ has finite tree-width.*

Now we are ready to prove the main result of Section 3. It can be seen as a converse of Seese's Theorem for graphs with a high degree of symmetry.

Theorem 3.10. *Let G be a Γ-labeled connected graph of bounded degree such that $\mathrm{Aut}(G)$ has only finitely many orbits on G. Then the following properties are equivalent:*

(1) $\mathrm{MSOTh}(G)$ *is decidable.*
(2) $\mathrm{ud}(G)$ *has finite tree-width.*
(3) $\mathrm{ud}(G)$ *can be M-triangulated for some constant M.*
(4) G is context-free.

[6] The results in [15] are stated for sets of finite graphs, but it is easy to see that the restriction to finite graphs is actually not crucial.

Proof. Since G is connected and of bounded degree, G must be countable. The implication (1) $\Rightarrow$ (2) is stated in Theorem 3.9, whereas the implication (4) $\Rightarrow$ (1) is shown in [45].

For (2) $\Rightarrow$ (3) assume that $\mathrm{ud}(G)$ has finite tree-width. Since any automorphism of G is also an automorphism of $\mathrm{ud}(G)$, the group $\mathrm{Aut}(\mathrm{ud}(G))$ has only finitely many orbits on $\mathrm{ud}(G)$. Hence, by Theorem 3.7, there exists a strong tree decomposition P of $\mathrm{ud}(G)$ of width $\leq b$ such that for all $W \in P$, $\mathrm{diam}_{\mathrm{ud}(G)}(W) \leq c$. Here b and c are fixed constants. Now consider a sequence

$$\pi = [v_0, v_1, \ldots, v_{m-1}, v_0]$$

of nodes $v_i \in V(G)$. Let $W_i \in P$ be such that $v_i \in W_i$. Assume that for all $0 \leq i < m$, either $W_i = W_{i+1}$ or $(W_i, W_{i+1}) \in E(\mathrm{ud}(G)/_P)$ (here and in the following, all subscripts are interpreted modulo m). Note that this implies that $d_{\mathrm{ud}(G)}(v_i, v_{i+1}) \leq 2c + 1$ for all $0 \leq i < m$ and that

$$T_\pi = (\mathrm{ud}(G)/_P)\restriction\{W_0, \ldots, W_{m-1}\}$$

is a finite subtree of the tree $\mathrm{ud}(G)/_P$. By induction on m, we will construct a $(2c + 1)$-triangulation of π. Thus, in particular every closed path of G can be $(2c + 1)$-triangulated which shows (3).

The following construction is quite similar to the proof of [4, Theorem 8]. The case that T_π consists of a single node is obvious, since this implies $d_{\mathrm{ud}(G)}(v_i, v_j) \leq c$ for all i, j. Thus, assume that T_π has at least two nodes. Let W be a leaf of T_π and let W' be the unique neighbor of W in T_π. Thus, there exists a subsequence $[v_i, v_{i+1}, \ldots, v_k]$ of π with $k - i \geq 2$, $v_i, v_k \in W'$, and $v_{i+1}, \ldots, v_{k-1} \in W$. Since $\mathrm{diam}_{\mathrm{ud}(G)}(W \cup W') \leq 2c+1$, we can find a $(2c+1)$-triangulation of the sequence $[v_i, v_{i+1}, \ldots, v_k, v_i]$. Moreover, by induction there exists a $(2c + 1)$-triangulation of $[v_0, \ldots, v_i, v_k, \ldots, v_{m-1}, v_0]$. By gluing these two triangulations along the side $[v_i, v_k]$ (note that $d_{\mathrm{ud}(G)}(v_i, v_k) \leq 2c+1$), we obtain a $(2c+1)$-triangulation of π.

It remains to prove (3) $\Rightarrow$ (4). We can assume that G is a rooted graph by choosing an arbitrary root in G. This allows to use the notations $G(v)$ and $\Delta(v)$, see the last paragraph in Section 3.3. If $\mathrm{ud}(G)$ can be M-triangulated for some constant M, then by the argument given in the proof of [45, Theorem 2.9] it follows that $\mathrm{diam}_{\mathrm{ud}(G)}(\Delta(v)) \leq 3 \cdot M$ for every $v \in V(G)$. Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be the orbits of $\mathrm{Aut}(G)$ on G, and choose $v_i \in \mathcal{O}_i$ for every i arbitrarily. Now, if $v \in V(G)$ is arbitrary, then for some $1 \leq i \leq n$ we find an automorphism of G that maps $\Delta(G)$ injectively into the sphere $S_{\mathrm{ud}(G)}(3 \cdot M, v_i)$. Hence, since every $G(v)$ is uniquely determined by $\Delta(v)$ and the set of those edges that connect nodes from $\Delta(v)$ with nodes from $G \setminus G(v)$, there exist only finitely many $G(v)$ that are pairwise not end-isomorphic.[7] $\square$

[7] The latter argument appears in the proof of [45, Theorem 2.9] for a vertex-transitive graph, i.e., a graph where $\mathrm{Aut}(G)$ has only one orbit on G.

Remark 3.11. Let $G = (V, E)$ be a graph and let P be the partition of V given by the orbits of $\text{Aut}(G)$ on G. In [56] it was shown that if G is context-free, then also $G/_P$ is context-free. Hence, a natural generalization of Theorem 3.10 would be the following statement: Let G be a connected graph of bounded degree such that the quotient graph $G/_P$ is context-free. Then G has a decidable MSO-theory if and only if G is context-free. But this is in fact false: Take $\mathbb{N}$ together with the successor relation and add to every number $m = \frac{1}{2}n(n+1)$ $(n \in \mathbb{N})$ a copy m' together with the edge (m, m'), whereas for every other number m we add two copies m' and m'' together with the edges (m, m') and (m, m''). The resulting graph is not context-free, but it has a decidable MSO-theory [29] (see also [45]) and $G/_P$ is context-free. This example also shows that the restriction to finitely many orbits in Theorem 3.10 is necessary.

Remark 3.12. By [66, Remark 2], we can add the following two equivalent properties to the list of properties in Theorem 3.10, see [66] for the definition.

- $\text{ud}(G)$ admits a uniformly spanning tree.
- All ends of $\text{ud}(G)$ have finite diameter.

The next theorem generalizes Theorem 3.10 to graphs that are not necessarily connected and countable. For graphs G, G_1, G_2 and a cardinal α, αG denotes the graph that consists of α many disjoint copies of G, and $G_1 + G_2$ denotes the disjoint union of G_1 and G_2.

Theorem 3.13. *Let G be a graph of bounded degree but arbitrary cardinality such that $\text{Aut}(G)$ has only finitely many orbits on G. Then $\text{MSOTh}(G)$ is decidable if and only if there exist finitely many context-free graphs $G_1, \ldots, G_n$ and cardinals $\alpha_1, \ldots, \alpha_n$ such that $G = \alpha_1 G_1 + \cdots + \alpha_n G_n$.*

Proof. First assume that $G = \alpha_1 G_1 + \cdots + \alpha_n G_n$, where G_i is context-free. Thus, $\text{MSOTh}(G_i)$ is decidable. With [58] (see also [32, 62]) we can deduce that also $\text{MSOTh}(G)$ is decidable. Now assume that $\text{MSOTh}(G)$ is decidable, and let G_i, $i \in J$, be the connected components of G. Since every G_i is of bounded degree and connected, all G_i are countable. Furthermore, since $\text{Aut}(G)$ has only finitely many orbits on G, there exist only finitely many pairwise non-isomorphic G_i. Thus, $G = \alpha_1 G_1 + \cdots + \alpha_n G_n$ for cardinals α_i. Moreover, also every $\text{Aut}(G_i)$ has only finitely many orbits on G_i. Thus, in order to prove that every G_i is context-free, it suffices by Theorem 3.10 to show that $\text{MSOTh}(G_i)$ is decidable for all $1 \leq i \leq n$.

The set of graphs $\{G_1, \ldots, G_n\}$ can be partitioned into classes $\mathcal{C}_1, \ldots, \mathcal{C}_m$ $(m \leq n)$ such that $\text{MSOTh}(G_i) = \text{MSOTh}(G_j)$ if and only if $G_i, G_j \in \mathcal{C}_k$ for some k. Thus, for each class $\mathcal{C}_k$, we can select an MSO-sentence ψ_k such that $G_i \models \psi_k$ if and only if $G_i \in \mathcal{C}_k$. Now we can reduce $\text{MSOTh}(G_i)$ to $\text{MSOTh}(G)$ as follows: Assume that $G_i \in \mathcal{C}_k$. Given an MSO-sentence ϕ, we construct the following MSO-sentence ϕ_k (recall that the existence of a finite undirected path between two nodes can be expressed in MSO-logic):

$$\phi_k \quad \equiv \quad \exists X \begin{Bmatrix} \forall x, y \in X : d_{\mathrm{ud}(G)}(x,y) < \infty \,\wedge \\ \forall x \in X \; \forall y \notin X : d_{\mathrm{ud}(G)}(x,y) = \infty \,\wedge \\ \psi_k^X \;\wedge\; \phi^X \end{Bmatrix}$$

Here, ψ_k^X denotes the formula that results from ψ_k by relativizing all quantifiers in ψ_k to the set of nodes X, and similarly for ϕ^X. Then $G_i \models \phi$ if and only if $G \models \phi_k$. □

4 Logic over Cayley-graphs

In this section we will characterize those finitely generated groups whose Cayley-graph has a decidable MSO-theory (resp. first-order theory).

4.1 Monadic second-order logic

Recall that a finitely generated group $\mathcal{G}$ is called context-free if the set of all words over the generators that represent the unit of $\mathcal{G}$ is a context-free language. In [45] it is shown that a finitely generated group is context-free if and only if its Cayley-graph (with respect to any generating set) is context-free. Together with Theorem 3.10 we can deduce the following result:

Corollary 4.1. *Let $\mathcal{G}$ be a finitely generated group. The following properties are equivalent:*

(1) $\mathrm{MSOTh}(\mathcal{C}(\mathcal{G}))$ *is decidable.*
(2) $\mathrm{ud}(\mathcal{C}(\mathcal{G}))$ *has finite tree-width.*
(3) $\mathcal{G}$ *is context-free.*

Remark 4.2. Recall that a problem is *elementarily decidable* if it can be solved in time $O(2^{\cdot^{\cdot^{2^n}}})$ where the height of this tower of exponents is constant. We remark that the complexity of the MSO-theory is non-elementary for any Cayley-graph of an infinite context-free group $\mathcal{G}$: the Cayley-graph $\mathcal{C}(\mathcal{G}) = (V, (E_a)_{a\in\Gamma})$ is of bounded degree and infinite. Hence it contains an infinite simple path $(u_i)_{i\in\mathbb{N}}$ such that $(u_i, u_j) \in E = \bigcup_{a\in\Gamma} E_a$ if and only if $i + 1 = j$. If U is any set of nodes of $\mathcal{C}(\mathcal{G})$, then U is such an infinite simple path if and only if there exists a node $u \in U$ such that the following holds: (i) $(u, v) \in E^*$ for every $v \in U$, (ii) for every $v \in U$ there exists exactly one $v' \in U$ with $(v, v') \in E$, (iii) for every $v \in U \setminus \{u\}$ there exists exactly one $v' \in U$ with $(v', v) \in E$, and (iv) there does not exist $v \in U$ with $(v, u) \in E$. All this can be expressed in MSO-logic. Hence, we obtain a reduction of $\mathrm{MSOTh}(\mathbb{N}, \leq)$ to $\mathrm{MSOTh}(\mathcal{C}(\mathcal{G}))$ which implies the result by [43].

Remark 4.3. If $\mathcal{G}$ is a finitely generated group such that the corresponding Cayley-graph has finite tree-width, then the corollary above implies that $\mathcal{G}$ is context-free. Hence, by [44], $\mathcal{G}$ is finitely presented. It seems to be hard to deduce this fact in a direct way.

Remark 4.4. A variant of the equivalence of (2) and (3) in Corollary 4.1 is stated in [4]: $\mathcal{G}$ is context-free if and only if there exists a (ordinary) tree decomposition (T, f) of $\mathcal{C}(\mathcal{G})$ of finite width such that for all $w \in V(T)$, the subgraph $\mathcal{C}(\mathcal{G})\restriction f(w)$ is connected (note that in contrast to our notation, in [4] such a tree decomposition is called strong).

In contrast to this, our considerations from Section 3.2 show that $\mathcal{G}$ is context-free if and only if $\mathcal{C}(G)$ has a strong tree decomposition P of finite width such that the diameter of the partition classes in P is uniformly bounded. We do not know whether the partition classes in P can be even assumed to be connected. Or more generally: Does every graph that satisfies the conditions of Theorem 3.7 have a strong tree decomposition P of finite width such that moreover every partition class in P is connected?

Further results on the geometric structure of context-free groups can be found in [18, 47, 55].

4.2 The method of Ferrante and Rackoff

Before we continue with the investigation of first-order theories of Cayley-graphs, we briefly interrupt for the discussion of a method of Ferrante and Rackoff for proving upper bounds on the complexity of first-order theories. If we want to test validity of a first-order sentence $\forall x\, \varphi$, we are faced with an infinite number of questions: $\varphi(x)$ has to be checked for all elements x of the underlying structure. Ferrante and Rackoff's method allows to identify finite sets such that checking $\varphi(x)$ for these elements suffices to infer $\forall x\, \varphi$. A first tool used here is Gaifman's locality theorem that we introduce first.

Let $\mathcal{A} = (A, (R_i)_{i\in J})$ be a relational structure where R_i has arity n_i. The *Gaifman-graph* $G_{\mathcal{A}}$ of the structure $\mathcal{A}$ is the following undirected graph:

$$G_{\mathcal{A}} = (A, \{(a,b) \in A \times A \mid \bigvee_{i\in J} \exists (c_1, \ldots, c_{n_i}) \in R_i\ \exists j,k : c_j = a \neq b = c_k\}).$$

We will mainly be interested in restrictions of the structure $\mathcal{A}$ to certain spheres in this graph. To ease notations, we will also write $S_{\mathcal{A}}(r, \widetilde{a})$ for $\mathcal{A}\restriction S_{G_{\mathcal{A}}}(r, \widetilde{a})$, i.e., $S_{\mathcal{A}}(r, \widetilde{a})$ is the substructure of $\mathcal{A}$ induced by the r-sphere around the tuple $\widetilde{a}$ in the Gaifman-graph of $\mathcal{A}$. Then $(S_{\mathcal{A}}(r, \widetilde{a}), \widetilde{a})$ is this substructure, where in addition all elements from the tuple $\widetilde{a}$ are added as constants.

Roughly speaking, Gaifman's Theorem [31] states that first-order logic only allows to express local properties of structures, see [28] for a recent account of this result. For our use, the following weaker form of Gaifman's Theorem is sufficient which is an immediate consequence of the main theorem in [31].

Theorem 4.5 (cf. [31]). *Let* $\widetilde{a} = (a_1, a_2, \ldots, a_k)$ *and* $\widetilde{b} = (b_1, b_2, \ldots, b_k)$*, where* $a_i, b_i \in A$*, such that*

$$(S_{\mathcal{A}}(7^n, \widetilde{a}), \widetilde{a}) \cong (S_{\mathcal{A}}(7^n, \widetilde{b}), \widetilde{b}).^{8}$$

Then, for any first-order formula $\varphi(x_1, x_2, \ldots, x_k)$ *of quantifier-depth at most* n*, we have*

$$\mathcal{A} \models \varphi(\widetilde{a}) \text{ if and only if } \mathcal{A} \models \varphi(\widetilde{b}).$$

Now we use this theorem to restrict the domain of quantification to elements of "small norm": a *norm function* on $\mathcal{A}$ is just a function $\lambda : A \to \mathbb{N}$. We write $\mathcal{A} \models \exists x \leq n : \varphi$ in order to express that there exists $a \in A$ such that $\lambda(a) \leq n$ and $\mathcal{A} \models \varphi(a)$, and similarly for $\forall x \leq n : \varphi$. One can indeed restrict quantification to small elements provided the structure in question is H-bounded (Ferrante and Rackoff [30]):

Definition 4.6. *Let* λ *be a norm function on* $\mathcal{A}$*. Let furthermore*

$$H : \{(j, d) \in \mathbb{N} \times \mathbb{N} \mid j \leq d\} \to \mathbb{N}$$

be a function such that the following holds: For any $j \leq d \in \mathbb{N}$*, any* $\widetilde{a} = (a_1, a_2, \ldots, a_{j-1}) \in A^{j-1}$ *with* $\lambda(a_i) \leq H(i, d)$*, and any* $a \in A$*, there exists* $a_j \in A$ *with* $\lambda(a_j) \leq H(j, d)$ *and*

$$(S_{\mathcal{A}}(7^{d-j}, \widetilde{a}, a), \widetilde{a}, a) \cong (S_{\mathcal{A}}(7^{d-j}, \widetilde{a}, a_j), \widetilde{a}, a_j).$$

Then $\mathcal{A}$ *is called* H-bounded *(with respect to the norm function* λ*).*

This is a slight variant of the definition in [30] that suits our needs much better than the original formulation. The following corollary to Theorem 4.5 was shown by Ferrante and Rackoff for their version of H-bounded structures.

Corollary 4.7 (cf. [30]). *Let* $\mathcal{A}$ *be a relational structure with norm* λ *and let* $H : \{(j, d) \in \mathbb{N} \times \mathbb{N} \mid j \leq d\} \to \mathbb{N}$ *be a function such that* $\mathcal{A}$ *is* H*-bounded. Then for any first-order sentence* $\varphi \equiv Q_1 x_1\, Q_2 x_2 \cdots Q_d x_d : \psi$ *where* ψ *is quantifier free and* $Q_i \in \{\exists, \forall\}$*, we have* $\mathcal{A} \models \varphi$ *if and only if*

$$\mathcal{A} \models Q_1 x_1 \leq H(1, d)\, Q_2 x_2 \leq H(2, d) \cdots Q_d x_d \leq H(d, d) : \psi.$$

Proof. For $j \leq d$, let $\psi_j \equiv Q_j x_j\, Q_{j+1} x_{j+1} \cdots Q_d x_d : \psi$ and

$$\varphi_j \;\equiv\; Q_1 x_1 \leq H(1, d) \cdots Q_{j-1} x_{j-1} \leq H(j-1, d) : \psi_j,$$

in particular, $\varphi_1 \equiv \varphi$. We show that $\mathcal{A} \models \varphi_j$ if and only if $\mathcal{A} \models \varphi_{j+1}$ which then proves the corollary.

[8] Thus, there exists a bijection $f : S_{\mathcal{A}}(7^n, \widetilde{a}) \to S_{\mathcal{A}}(7^n, \widetilde{b})$ which preserves and reflects all relations from $\mathcal{A}$ and such that $f(a_i) = b_i$ for $1 \leq i \leq k$.

Let $\widetilde{a} = (a_1, \ldots, a_{j-1}) \in A^{j-1}$ with $\lambda(a_i) \leq H(i,d)$. First assume $Q_j = \exists$, i.e., $\psi_j \equiv \exists x_j : \psi_{j+1}$. If $\mathcal{A} \models \psi_j(\widetilde{a})$, then there is $a \in A$ with $\mathcal{A} \models \psi_{j+1}(\widetilde{a}, a)$. By our assumption on the norm function λ, we find $a_j \in A$ with $\lambda(a_j) \leq H(j,d)$ and

$$(S_{\mathcal{A}}(7^{d-j}, \widetilde{a}, a), \widetilde{a}, a) \cong (S_{\mathcal{A}}(7^{d-j}, \widetilde{a}, a_j), \widetilde{a}, a_j). \tag{1}$$

Since the quantifier depth of ψ_{j+1} is $d - j$, Theorem 4.5 implies $\mathcal{A} \models \psi_{j+1}(\widetilde{a}, a_j)$ and therefore $\mathcal{A} \models (\exists x_j \leq H(j,d)\, \psi_{j+1})(\widetilde{a})$. If, conversely,

$$\mathcal{A} \models (\exists x_j \leq H(j,d)\, \psi_{j+1})(\widetilde{a}),$$

we have trivially $\mathcal{A} \models \psi_j(\widetilde{a})$.

Assume now that $Q_j = \forall$, i.e., $\psi_j \equiv \forall x_j : \psi_{j+1}$. If $\mathcal{A} \models \psi_j(\widetilde{a})$, then of course also $\mathcal{A} \models (\forall x_j \leq H(j,d) : \psi_{j+1})(\widetilde{a})$. Now assume that

$$\mathcal{A} \models (\forall x_j \leq H(j,d) : \psi_{j+1})(\widetilde{a})$$

and let $a \in A$ be arbitrary. We have to show that $\mathcal{A} \models \psi_{j+1}(\widetilde{a}, a)$. The case $\lambda(a) \leq H(j,d)$ is clear. Thus, assume that $\lambda(a) > H(j,d)$. Then there exists $a_j \in A$ with $\lambda(a_j) \leq H(j,d)$ and (1). Since $\lambda(a_j) \leq H(j,d)$, we have $\mathcal{A} \models \psi_{j+1}(\widetilde{a}, a_j)$. Finally, Theorem 4.5 implies $\mathcal{A} \models \psi_{j+1}(\widetilde{a}, a)$. □

4.3 First-order logic

Let us now consider first-order theories of Cayley-graphs of groups.

Using the method of Ferrante and Rackoff, we start this section by proving an upper bound for the complexity of the first-order theory of the Cayley-graph in terms of the complexity of the word problem. The idea is to show that the Cayley-graph in question is H-bounded for some suitable norm λ and function H. Since this allows to restrict quantifications to finitely many elements, one can than exhaustively search for witnesses of the formula. Due to quantifier alternations, the resulting upper bound is best expressed in terms of alternating complexity classes. Let $\mathrm{ATIME}(a(n), t(n))$ (where $a(n) \leq t(n)$) denote the class of all problems that can be solved on an alternating Turing-machine in time $O(t(n))$ with at most $O(a(n))$ alternations [13, 46].

Theorem 4.8. *Let $\mathcal{G}$ be a finitely generated group such that the word problem of $\mathcal{G}$ belongs to the class* $\mathrm{ATIME}(a(n), t(n))$. *Then* $\mathrm{FOTh}(\mathcal{C}(\mathcal{G}))$ *belongs to* $\mathrm{ATIME}(n + a(2^{O(n)}), 2^{O(n)} + t(2^{O(n)}))$.

Proof. Choose a finite generating set Γ for $\mathcal{G}$. We want to apply Corollary 4.7, which requires to define the norm function λ and the bounding function H. For a group element $a \in \mathcal{G}$ let $\lambda(a) \in \mathbb{N}$ denote the least number n such that there exists a word $w \in \Gamma^*$ of length n, representing a. Thus, $\lambda(a)$ is the minimal length of a path from the identity 1 to a in the Cayley-graph $\mathcal{C} = \mathcal{C}(\mathcal{G})$. Next we define the function H by $H(j,d) = H(j-1,d) + 4 \cdot 7^{d-j}$ for $1 \leq j \leq d$ and set $H(0,d) = 0$. Thus, $H(j,d) \in 2^{O(d)}$.

Now let $j \leq d$ and $\widetilde{a} = (a_1, a_2, \dots, a_{j-1}) \in \mathcal{G}^{j-1}$ with $\lambda(a_i) \leq H(i,d)$. Let furthermore $a \in \mathcal{G}$ with $\lambda(a) > H(j,d)$. The triangle inequality implies that the distance between a and every a_i in $\mathcal{C}$ is larger than $H(j,d) - H(i,d) \geq H(j,d) - H(j-1,d) = 4 \cdot 7^{d-j}$ for $i < j$. Hence, $S_{\mathcal{C}}(7^{d-j}, \widetilde{a}) \cap S_{\mathcal{C}}(7^{d-j}, a) = \emptyset$ and moreover there is no edge in the graph $\mathcal{C}$ between a node in $S_{\mathcal{C}}(7^{d-j}, \widetilde{a})$ and a node in $S_{\mathcal{C}}(7^{d-j}, a)$.

Now assume that $a_j \in \mathcal{G}$ is any group element with $\lambda(a_j) = H(j,d)$. Since $\mathrm{Aut}(\mathcal{C})$ has only one orbit on $\mathcal{C}$, we have $(S_{\mathcal{C}}(7^{d-j}, a), a) \cong (S_{\mathcal{C}}(7^{d-j}, a_j), a_j)$. Moreover, $\lambda(a_j) = H(j,d)$ implies that also $S_{\mathcal{C}}(7^{d-j}, \widetilde{a}) \cap S_{\mathcal{C}}(7^{d-j}, a_j) = \emptyset$, and that there are no edges between these two disjoint spheres. It follows that

$$(S_{\mathcal{C}}(7^{d-j}, \widetilde{a}, a), \widetilde{a}, a) \cong (S_{\mathcal{C}}(7^{d-j}, \widetilde{a}, a_j), \widetilde{a}, a_j).$$

Thus, indeed, the Cayley-graph $\mathcal{C}$ is H-bounded.

Let $\varphi \equiv Q_1 x_1 Q_2 x_2 \cdots Q_d x_d : \psi(x_1, \dots, x_d)$ be a first-order sentence over the signature of $\mathcal{C}$ with d quantifiers $Q_i \in \{\exists, \forall\}$. Then, by Corollary 4.7, $\mathcal{C} \models \varphi$ if and only if

$$\mathcal{C} \models Q_1 x_1 \leq H(1,d)\, Q_2 x_2 \leq H(2,d) \cdots Q_d x_d \leq H(d,d) : \psi(x_1, \dots, x_d).$$

Since $H(i,d) \in 2^{O(|\varphi|)}$, this implies the statement of the theorem: In order to verify the above statement, an alternating Turing machine guesses for $1 \leq i \leq d$ words $w_i \in \Gamma^*$ with $|w_i| \leq H(i,d)$. If $Q_i = \exists$ (resp. $Q_i = \forall$), then the guessing is done in an existential (resp. universal) state of the alternating machine. Every quantifier alternation leads to one additional alternation, which leads to at most $|\varphi|$ alternations. After having guessed every w_i, we check whether $\psi(w_1, \dots, w_d)$ is true in the group $\mathcal{G}$. All identities in $\psi(w_1, \dots, w_d)$ have at most exponential length. These identities can be verified using the $\mathrm{ATIME}(a(n), s(n))$-algorithm for the word problem. This leads to $a(2^{O(|\varphi|)})$ many additional alternations. The time bound from the theorem follows analogously. □

It is known that $\mathrm{ATIME}(t(n), t(n))$ is contained in $\mathrm{DSPACE}(t(n))$ if $t(n) \geq n$ [13, Theorem 3.2]. Hence, differently from the situation for monadic second-order logic, the first-order theory is elementarily decidable as soon as the word problem is elementarily decidable. This is in particular the case for context-free groups. Theorem 4.8 allows us to prove the following result.

Theorem 4.9. *Let $\mathcal{G}$ be a finitely generated group. Then the following properties are equivalent:*

(1) $\exists\mathrm{FOTh}(\mathcal{C}(\mathcal{G}))$ *is decidable.*
(2) $\mathrm{FOTh}(\mathcal{C}(\mathcal{G}))$ *is decidable.*
(3) The word problem of $\mathcal{G}$ is decidable.

Proof. (2) ⇒ (1) is trivial, whereas (3) ⇒ (2) follows from Theorem 4.8. In order to prove (1) ⇒ (3), choose a finite generating set Γ for $\mathcal{G}$. Then a word $a_0 \cdots a_{n-1}$ ($a_i \in \Gamma$) represents the identity in $\mathcal{G}$ if and only if the following sentence belongs to $\exists\mathrm{FOTh}(\mathcal{C}(\mathcal{G}))$:

$$\exists x_0 \cdots \exists x_n \left\{ \bigwedge_{0 \le i \le n-1} x_i a_i = x_{i+1} \;\wedge\; x_0 = x_n \right\}$$

□

Remark 4.10. Note that the reductions of the word problem to the existential theory of the Cayley-graph and that of the existential theory to the full first-order theory are linear time logspace reductions. For the complexity of the full first-order theory in terms of the complexity of the word problem, see Theorem 4.8.

Remark 4.11. An alternative proof for the implication (3)⇒(2) in Theorem 4.9 can be given using a quantifier elimination procedure from [61]. But this procedure does not yield an elementary upper bound for the complexity of the first-order theory in terms of the complexity of the word problem.

The final result of this paper characterizes those groups such that the corresponding Cayley-graph has recursively enumerable positive first-order theory. Note that the full first-order theory of a Cayley-graph is recursively enumerable if and only if it is decidable, because it is a complete theory.

Theorem 4.12. *Let $\mathcal{G}$ be a finitely generated group. Then the following properties are equivalent:*

(1) $\mathrm{posFOTh}(\mathcal{C}(\mathcal{G}))$ *is recursively enumerable.*
(2) The word problem of $\mathcal{G}$ is recursively enumerable.

Proof. Let Γ be a finite set of generators for $\mathcal{G} = (G, \circ, 1)$. Assume w.l.o.g. that $\mathcal{G}$ is an infinite group. Let $h : \Gamma^* \to \mathcal{G}$ be the canonical homomorphism. If $\mathrm{posFOTh}(\mathcal{C}(\mathcal{G}))$ is recursively enumerable, then the proof of (1)⇒(3) in Theorem 4.9 shows that $W(\mathcal{G}, \Gamma)$ is recursively enumerable. Now assume that $W(\mathcal{G}, \Gamma)$ is recursively enumerable. In order to show that $\mathrm{posFOTh}(\mathcal{C}(\mathcal{G}))$ is recursively enumerable, we basically apply the quantifier elimination procedure from [61]. For this, we have to allow atomic predicates of the form $y \circ h(w) = x$ and $h(w) = 1$ for $w \in \Gamma^*$ and variables x and y. We will write briefly $yw = x$ and $w = 1$, respectively for these atomic predicates. Note that $yw = x$ is equivalent to $xw^{-1} = y$ and that $xw = x$ is equivalent to $w = 1$.

Now assume that we have given a positive formula of the form

$$\exists x \left\{ \bigwedge_{i=1}^{m} y_i u_i = x \;\wedge\; \varphi \right\},$$

where $m \geq 1$ and φ is a positive Boolean combination of atomic predicates that do not contain the variable x. Then this formula is equivalent to

$$\bigwedge_{i=2}^{m} y_i u_i u_1^{-1} = y_1 \ \wedge \ \varphi.$$

If we have given a positive formula of the form

$$\forall x \left\{ \bigvee_{i=1}^{m} y_i u_i = x \ \vee \ \varphi \right\},$$

where m and φ are as above, then, since $\mathcal{G}$ is infinite, this formula is equivalent to φ.

By applying the previous two reduction steps and switching between disjunctive normal form and conjunctive normal form for every quantifier alternation, we can effectively reduce every positive formula to an equivalent formula of the form

$$\bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} w_{i,j} = 1,$$

where $w_{i,j} \in \Gamma^*$. By enumerating $W(\mathcal{G}, \Gamma)$, it is clear that we can enumerate all true statements of this form. □

5 Open problems

Within the class of "symmetric" graphs of bounded degree, we gave a complete characterization of those having a decidable MSO-theory. Is a similar characterization possible within the class of all "symmetric" graphs (not necessarily of bounded degree)? Suppose the MSO-theory of $G^{(e)}$ is decidable. Then, by Seese's Theorem 3.8, the graph G has finite tree-width. Hence it can be described by an infinite term over the appropriate signature [14]. It is well possible that this term is quite regular provided the graph is "symmetric". If this is indeed the case, then the term (seen as a tree) and therefore the structure $G^{(e)}$ can be interpreted in the complete binary tree. This would imply that G is equational [14]. Since the MSO-theory of $G^{(e)}$ is decidable whenever G is equational, this would characterize this class of graphs. If, what is weaker, the MSO-theory of G is decidable, adopting Seese's conjecture [54], the graph G has finite clique-width [17]. This could, as above, imply that G can be interpreted in the complete binary tree, i.e., that G is prefix-recognizable [9]. This would characterize the "symmetric" graphs with a decidable MSO-theory.

References

1. I. Adler, H. Scheuermann, and M. Weyer. Finite character of f-tree-width via logical compactness. Manuscript.

2. A. V. Anisimov. Group languages. *Kibernetika*, 4:18–24, 1971. In Russian; English translation in: *Cybernetics 4*, 594–601, 1973.
3. L. Babai. Automorphism groups, isomorphism, reconstruction. In R. L. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of Combinatorics*, volume II, chapter 27, pages 1447–1540. Elsevier Science Publishers, 1995.
4. A. Blumensath. Prefix-recognizable graphs and monadic second-order logic. Technical Report 2001-06, RWTH Aachen, Department of Computer Science, 2001.
5. H. L. Bodlaender. A note on domino treewidth. *Discrete Mathematics & Theoretical Computer Science*, 3(4):141–150, 1999.
6. H. L. Bodlaender and J. Engelfriet. Domino treewidth. *Journal of Algorithms*, 24(1):94–123, 1997.
7. W. W. Boone and G. Higman. An algebraic characterization of groups with soluble word problem. *Journal of the Australian Mathematical Society*, 18:41–53, 1974.
8. D. Caucal. On the regular structure of prefix rewriting. *Theoretical Computer Science*, 106:61–86, 1992.
9. D. Caucal. On infinite transition graphs having a decidable monadic theory. In F. M. auf der Heide and B. Monien, editors, *Proceedings of the 23nd International Colloquium on Automata, Languages and Programming (ICALP'96), Paderborn (Germany)*, number 1099 in Lecture Notes in Computer Science, pages 194–205. Springer, 1996.
10. D. Caucal. On infinite terms having a decidable monadic theory. In K. Diks and W. Rytter, editors, *Proceedings of the 27th International Symposium on Mathematical Foundations of Computer Science (MFCS 2002), Warsaw (Poland)*, number 2420 in Lecture Notes in Computer Science, pages 165–176. Springer, 2002.
11. D. Caucal. On infinite transition graphs having a decidable monadic theory. *Theoretical Computer Science*, 290(1):79–115, 2002.
12. A. Cayley. On the theory of groups. *Proceedings of the London Mathematical Society (1)*, 9:126–133, 1878.
13. A. K. Chandra, D. C. Kozen, and L. J. Stockmeyer. Alternation. *Journal of the Association for Computing Machinery*, 28(1):114–133, 1981.
14. B. Courcelle. The monadic second-order logic of graphs, II: Infinite graphs of bounded width. *Mathematical Systems Theory*, 21:187–221, 1989.
15. B. Courcelle. The monadic second-order logic of graphs VI: On several representations of graphs by relational structures. *Discrete Applied Mathematics*, 54:117–149, 1994.
16. B. Courcelle. The expression of graph properties and graph transformations in monadic second-order logic. In G. Rozenberg, editor, *Handbook of graph grammars and computing by graph transformation, Volume 1 Foundations*, pages 313–400. World Scientific, 1997.
17. B. Courcelle and S. Olari. Upper bounds on the clique-width of graphs. *Discrete Applied Mathematics*, 101:77–114, 2000.
18. G. D'Agostino. Cayley graphs of virtually free groups. *International Journal of Algebra and Computation*, 3(2):189–199, 1993.
19. M. Dehn. Über die Toplogie des dreidimensionalen Raumes. *Mathematische Annalen*, 69:137–168, 1910. In German.
20. W. Dicks and M. J. Dunwoody. *Groups Acting on Graphs*. Cambridge University Press, 1989.
21. V. Diekert and M. Lohrey. A note on the existential theory of equations in plain groups. *International Journal of Algebra and Computation*, 12(1 & 2):1–7, 2002.
22. V. Diekert and M. Lohrey. Word equations over graph products. In P. K. Pandya and J. Radhakrishnan, editors, *Proceedings of the 23rd Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2003), Mumbai (India)*, number 2914 in Lecture Notes in Computer Science, pages 156–167. Springer, 2003.
23. V. Diekert and M. Lohrey. Existential and positive theories of equations in graph products. *Theory of Computing Systems*, 37(1): 133–156, 2004.
24. R. Diestel. *Graph Theory, Second Edition*. Springer, 2000.
25. G. Ding and B. Oporowski. Some results on tree decomposition of graphs. *Journal of Graph Theory*, 20:481–499, 1995.
26. M. J. Dunwoody. Cutting up graphs. *Combinatorica*, 2(1):15–23, 1981.
27. M. J. Dunwoody. The accessibility of finitely presented groups. *Inventiones Mathematicae*, 81:449–457, 1985.

28. H.-D. Ebbinghaus and J. Flum. *Finite Model Theory*. Springer, 1991.
29. C. C. Elgot and M. O. Rabin. Decidability and undecidability of extensions of second (first) order theory of (generalized) successor. *Journal of Symbolic Logic*, 31(2):169–181, 1966.
30. J. Ferrante and C. Rackoff. *The Computational Complexity of Logical Theories*. Number 718 in Lecture Notes in Mathematics. Springer, 1979.
31. H. Gaifman. On local and nonlocal properties. In J. Stern, editor, *Logic Colloquium '81*, pages 105–135. North Holland, 1982.
32. Y. Gurevich. Monadic second-order theories. In J. Barwise and S. Feferman, editors, *Model-Theoretic Logics*, pages 479–506. Springer, 1985.
33. T. Herbst and R. M. Thomas. Group presentations, formal languages and characterizations of one-counter groups. *Theoretical Computer Science*, 112:187–213, 1983.
34. G. Higman. Subgroups of finitely presented groups. *Proceedings of the Royal Society of London, Series A*, 262:455–475, 1961.
35. W. Hodges. *Model Theory*. Cambridge University Press, 1993.
36. D. Kuske and M. Lohrey. Decidable theories of Cayley-graphs. In H. Alt and M. Habib, editors, *Proceedings of the 20th Annual Symposium on Theoretical Aspects of Computer Science (STACS 2003), Berlin (Germany)*, number 2607 in Lecture Notes in Computer Science, pages 463–474. Springer, 2003.
37. A. A. Letičevskiĭ and L. B. Smikun. On a class of groups with solvable problem of automata equivalence. *Dokl. Akad. Nauk SSSR*, 227:36–38, 1976. In Russian; English translation in *Sov. Math., Dokl. 17*, 341-344, 1976.
38. R. C. Lyndon. On Dehn's algorithm. *Mathematical Annales*, 166:208–228, 1966.
39. R. C. Lyndon and P. E. Schupp. *Combinatorial Group Theory*. Springer, 1977.
40. W. Magnus, A. Karrass, and D. Solitar. *Combinatorial Group Theory*. Wiley, 1966.
41. G. S. Makanin. Equations in a free group. *Izv. Akad. Nauk SSR*, Ser. Math. 46:1199–1273, 1983. In Russian; English translation in *Math. USSR Izvestija 21, 1983*.
42. G. S. Makanin. Decidability of the universal and positive theories of a free group. *Izv. Akad. Nauk SSSR*, Ser. Mat. 48:735–749, 1984. In Russian; English translation in: *Math. USSR Izvestija, 25*, 75–88, 1985.
43. A. R. Meyer. Weak monadic second order theory of one successor is not elementary recursive. In *Proceedings of the Logic Colloquium (Boston 1972–73)*, number 453 in Lecture Notes in Mathematics, pages 132–154. Springer, 1975.
44. D. E. Muller and P. E. Schupp. Groups, the theory of ends, and context-free languages. *Journal of Computer and System Sciences*, 26:295–310, 1983.
45. D. E. Muller and P. E. Schupp. The theory of ends, pushdown automata, and second-order logic. *Theoretical Computer Science*, 37(1):51–75, 1985.
46. C. H. Papadimitriou. *Computational Complexity*. Addison Wesley, 1994.
47. L. Pélecq. Automorphism groups of context-free graphs. *Theoretical Computer Science*, 165(2):275–293, 1996.
48. M. O. Rabin. Decidability of second-order theories and automata on infinite trees. *Transactions of the American Mathematical Society*, 141:1–35, 1969.
49. E. Rips and Z. Sela. Canonical representatives and equations in hyperbolic groups. *Inventiones Mathematicae*, 120:489–512, 1995.
50. N. Robertson and P. D. Seymour. Graph minors II. Algorithmic aspects of tree-width. *Journal of Algorithms*, 78:309–322, 1986.
51. P. E. Schupp. Groups and graphs: Groups acting on trees, ends, and cancellation diagrams. *Mathematical Intelligencer*, 1:205–222, 1979.
52. P. E. Schupp. Arrays, automata and groups: some interconnections. In Ch. Choffrut, editor, *Proceedings of Automata Networks, LITP Spring School on Theoretical Computer Science, Angelés-Village (France)*, number 316 in Lecture Notes in Computer Science, pages 19–28. Springer, 1988.
53. D. Seese. Tree-partite graphs and the complexity of algorithms. In L. Budach, editor, *Proceedings of Fundamentals of Computation Theory (FCT'85), Cottbus (GDR)*, number 199 in Lecture Notes in Computer Science, pages 412–421, 1985.
54. D. Seese. The structure of models of decidable monadic theories of graphs. *Annals of Pure and Applied Logic*, 53:169–195, 1991.

55. G. Sénizergues. An effective version of Stalling's theorem in the case of context-free groups. In A. Lingas, R. G. Karlsson, and S. Carlsson, editors, *Proceedings of the 20th International Colloquium on Automata, Languages and Programming (ICALP 93), Lund (Sweden)*, number 700 in Lecture Notes in Computer Science, pages 478–495. Springer, 1993.
56. G. Sénizergues. Semi-groups acting on context-free graphs. In F. M. auf der Heide and B. Monien, editors, *Proceedings of the 23th International Colloquium on Automata, Languages and Programming (ICALP'96), Paderborn (Germany)*, number 1099 in Lecture Notes in Computer Science, pages 206–218. Springer, 1996.
57. J.-P. Serre. *Trees*. Springer, 1980.
58. S. Shelah. The monadic theory of order. *Annals of Mathematics, II. Series*, 102:379–419, 1975.
59. L. B. Smikun. Über den Zusammenhang kontextfreier Gruppen und Gruppen mit lösbarem Problem der Äquivalenz von Automaten. *Kibernetika*, 5:33–37, 1976. In Russian.
60. J. R. Stallings. *Group Theory and Three-Dimensional Manifolds*. Number 4 in Yale Mathematical Monographs. Yale University Press, 1971.
61. A. P. Stolboushkin and M. A. Taitslin. Deterministic dynamic logic is strictly weaker than dynamic logic. *Information and Control*, 57(1):48–55, 1983.
62. W. Thomas. Ehrenfeucht games, the composition method, and the monadic theory of ordinal words. In J. Mycielski, G. Rozenberg, and A. Salomaa, editors, *Structures in Logic and Computer Science, A Selection of Essays in Honor of A. Ehrenfeucht*, number 1261 in Lecture Notes in Computer Science, pages 118–143. Springer, 1997.
63. W. Thomas. Elements of an automata theory over partial orders. In D. Peled et al., editors, *Partial order methods in verification*, pages 25–40. DIMACS Series in Discrete Mathematics and Theoretical Computer Science vol. 29, 1997.
64. C. Thomassen. Configurations in graphs of large minimum degree, connectivity, or chromatic number. In *Proceedings of the 3rd International Conference on Combinatorial Mathematics*, number 555 in Annals of the New York Academy of Sciences, pages 402–412, 1989.
65. C. Thomassen and W. Woess. Vertex-transitive graphs and accessibility. *Journal of Combinatorial Theory, Series B*, 58:248–268, 1993.
66. W. Woess. Graphs and groups with tree-like properties. *Journal of Combinatorial Theory, Series B*, 47:361–371, 1989.