# The submonoid and rational subset membership problems for graph groups

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**Abstract.** It is shown that the membership problem in a finitely generated submonoid of a graph group (also called a right-angled Artin group or a free partially commutative group) is decidable if and only if the independence graph (commutation graph) is a transitive forest. In particular, we obtain the first example of a finitely presented group with a decidable generalized word problem that does not have a decidable membership problem for finitely generated submonoids. We also show that the rational subset membership problem is decidable for a graph group if and only if the independence graph is a transitive forest. This answers a question of Kambites, Silva, and the second author [25].

## 1 Introduction

Algorithmic problems concerning groups are a classical topic in algebra and theoretical computer science. Since the pioneering work of Dehn from 1910 [10], decision problems like the word problem or the generalized word problem (which is also known as the subgroup membership problem since it asks whether one can decide if a given group element belongs to a given finitely generated subgroup) have been intensively studied for various classes of groups. A first natural generalization of these classical decision problems is the submonoid membership problem: given a finite set $S$ of elements of $G$ and an element $g \in G$, can one decide whether $g$ belongs to the submonoid generated by $S$. Notice that $g$ has finite order if and only if $g^{-1}$ is in the submonoid generated by $g$ and so decidability of the submonoid membership problem lets one determine algorithmically the order of an element of the group $G$. A recent paper on the submonoid membership problem is Margolis, Meakin and Šuniḱ [28].

A further generalization is the rational subset membership problem: for a given rational subset $L$ of a group $G$ and an element $g \in G$ it is asked whether $g \in L$. The class of rational subsets of a group $G$ is the smallest class that contains all finite subsets of $G$, and which is closed under union, product, and the Kleene hull (or Kleene star; it associates to a subset $L \subseteq G$ the submonoid $L^*$ generated by $L$). Rational subsets in arbitrary groups and monoids are an important research topic in language theory, see, e.g., [3]. The rational subset membership problem generalizes the submonoid membership problem and the the generalized word problem for a group, because every finitely generated submonoid (and hence subgroup) of a group is rational.

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It is easy to see that decidability of the rational subset membership problem transfers to finitely generated subgroups. Grunschlag has shown that the property of having a decidable rational subset membership problem is preserved under finite extensions, i.e., if $G$ has a decidable rational subset membership problem and $G \leq H$, where the index of $G$ in $H$ is finite, then $H$ also has a decidable rational subset membership problem [21]. Kambites, Silva, and the second author [25] proved that the fundamental group of a finite graph of groups [36] with finite edge groups has a decidable rational subset membership problem provided all vertex groups have a decidable rational subset membership problem. In particular, this implies that decidability of the rational subset membership problem is preserved by free products, see also [31].

The main result of this paper is to characterize the decidability of the submonoid membership problem and the rational subset membership problem for graph groups. In particular we provide the first example, as far as we know, of a group with a decidable generalized word problem that does not have a decidable submonoid (and hence rational subset) membership problem.

A *graph group* [15] $\mathbb{G}(\Sigma, I)$ is specified by a finite undirected graph $(\Sigma, I)$, which is also called an *independence alphabet* (or *commutation graph*). The graph group $\mathbb{G}(\Sigma, I)$ is formally defined as the quotient group of the free group generated by $\Sigma$ modulo the set of all relations $ab = ba$, where $(a, b) \in I$. Graph groups are a group analogue to trace monoids (free partially commutative monoids), which play a prominent role in concurrency theory [14]. Graph groups are also called *free partially commutative groups* [12, 38], *right-angled Artin groups* [7, 9], and *semifree groups* [2]. They are currently a hot topic of interest in group theory, in particular because of the richness of the class of groups embeddable in graph groups. For instance, the Bestvina-Brady groups, which were used to distinguish the finiteness properties $\mathcal{F}_n$ and $\mathrm{FP}_n$ [4] (and were also essential for distinguishing the finiteness properties FDT and FHT for string rewriting systems [33]), are subgroups of graph groups. Crisp and Wiest show that the fundamental group of any orientable surface (and of most non-orientable surfaces) embeds in a graph group [9].

Algorithmic problems concerning graph groups have been intensively studied in the past, see, e.g., [12, 13, 17, 25, 26, 38]. In [12, 38] it was shown that the word problem for a graph group can be decided in linear time (on a random access machine). A recent result of Kapovich, Weidmann, and Myasnikov [26] shows that if $(\Sigma, I)$ is a chordal graph (i.e., if $(\Sigma, I)$ does not have an induced cycle of length at least 4), then the generalized word problem for $\mathbb{G}(\Sigma, I)$ is decidable. On the other hand, a classical result of Mihailova [30] states that already the generalized word problem for the direct product of two free groups of rank 2 is undecidable. Note that this group is the graph group $\mathbb{G}(\Sigma, I)$, where the graph $(\Sigma, I)$ is a cycle on 4 nodes (also called $\mathsf{C4}$). In fact, Mihailova proves a stronger result: she constructs a *fixed* subgroup $H$ of $\mathbb{G}(\mathsf{C4})$ such that it is undecidable, whether a given element of $\mathbb{G}(\mathsf{C4})$ belongs to $H$. Recently, it was shown by Kambites that a graph group $\mathbb{G}(\Sigma, I)$ contains a direct product of two free groups of rank 2 if and only if $(\Sigma, I)$ contains an induced $\mathsf{C4}$ [24]. This leaves a gap between the decidability result of [26] and the undecidability result of Mihailova [30].

In [25] it is shown that the rational subset membership problem is decidable for a free product of direct products of a free group with a free Abelian group. Such a group is a graph group $\mathbb{G}(\Sigma, I)$, where every connected component of $(\Sigma, I)$ results from connecting all nodes of a clique with all nodes from an edge-free graph. On the other hand, the only undecidability result for the rational subset membership problem for graph groups that was known so far is Mihailova's result for independence alphabets containing an induced C4.

In this paper, we shall characterize those graph groups for which the rational subset membership problem is decidable: we prove that these are exactly those graph groups $\mathbb{G}(\Sigma, I)$, where $(\Sigma, I)$ is a transitive forest (Theorem 2). The graph $(\Sigma, I)$ is a transitive forest if it is the disjoint union of comparability graphs of rooted trees. An alternative characterization of transitive forests was presented in [37]: $(\Sigma, I)$ is a transitive forest if and only if it neither contains an induced C4 nor an induced path on 4 nodes (also called P4). Graph groups $\mathbb{G}(\Sigma, I)$, where $(\Sigma, I)$ is a transitive forest, have also appeared in [29]: they are exactly those graph groups which are subgroup separable (the case of P4 appears in [32]. Recall that a group $G$ is called subgroup separable if, for every finitely generated subgroup $H \leq G$ and every $g \in G \setminus H$ there exists a normal subgroup $N \leq G$ having finite index such that $H \leq N$ and $g \notin N$. Subgroup separability implies decidability of the generalized word problem.

One half of Theorem 2 can be easily obtained from a result of Aalbersberg and Hoogeboom [1]: The problem of deciding whether the intersection of two rational subsets of the trace monoid (free partially commutative monoid) $\mathbb{M}(\Sigma, I)$ is nonempty is decidable if and only if $(\Sigma, I)$ is a transitive forest. Now, $L \cap K \neq \emptyset$ for two given rational subsets $L, K \subseteq \mathbb{M}(\Sigma, I)$ if and only if $1 \in LK^{-1}$ in the graph group $\mathbb{G}(\Sigma, I)$. Hence, if $(\Sigma, I)$ is not a transitive forest, then the rational subset membership problem for $\mathbb{G}(\Sigma, I)$ is undecidable. In fact, we construct a fixed rational subset $L \subseteq \mathbb{G}(\Sigma, I)$ such that it is undecidable, whether $g \in L$ for a given group element $g \in \mathbb{G}(\Sigma, I)$.

The converse direction in Theorem 2 is an immediate corollary of our Theorem 1, which is one of the main group theoretic results of this paper. It states that the rational subset membership problem is decidable for every group that can be built up from the trivial group using the following four operations: (i) taking finitely generated subgroups, (ii) finite extensions, (iii) direct products with $\mathbb{Z}$, and (iv) finite graphs of groups with finite edge groups. Note that the only operation that is not covered by the results cited earlier is the direct product with $\mathbb{Z}$. In fact, it seems to be an open question whether decidability of the rational subset membership problem is preserved under direct products with $\mathbb{Z}$. Hence, we have to follow another strategy. We will introduce a property of groups that implies the decidability of the rational subset membership problem, and which has all the desired closure properties. Our proof of Theorem 1 uses mainly techniques from formal language theory (e.g., semilinear sets, Parikh's theorem) and is inspired by the methods from [1, 6].

It should be noted that due to the above reduction from the intersection problem for rational trace languages to the rational subset membership problem for the corresponding graph group, we also obtain an alternative to the quite difficult proof from [1] for the implication "$(\Sigma, I)$ is a transitive forest $\Rightarrow$ intersection problem for rational subsets of $\mathbb{M}(\Sigma, I)$ is decidable".

In Section 4 we consider the *submonoid membership problem* for groups. We prove that the rational subset membership problem for a group $G$ can be reduced to the submonoid membership problem for the free product $G * \mathbb{Z}$ (Theorem 4). The idea is to encode the state of a finite automaton into the $\mathbb{Z}$-component of the free product $G * \mathbb{Z}$. Using similar techniques, we are also able to prove that the submonoid membership problem is undecidable for the graph group $\mathbb{G}(\Sigma, I)$, where $(\Sigma, I)$ is $\mathsf{P4}$ (Theorem 5). The result of [26] shows that this graph group does have a decidable generalized word problem, thereby giving our example of a group with a decidable generalized word problem but an undecidable submonoid membership problem. Together with Mihailova's undecidability result for $\mathsf{C4}$ and our decidability result for transitive forests (Theorem 2) it also follows that the submonoid membership problem for a graph group $\mathbb{G}(\Sigma, I)$ is decidable if and only if $(\Sigma, I)$ is a transitive forest (Corollary 1).

## 2 Preliminaries

We assume that the reader has some basic knowledge in formal language theory (see, e.g., [3, 23]) and group theory (see, e.g., [27, 35]).

### 2.1 Formal languages

Let $\Sigma$ be a finite alphabet. We use $\Sigma^{-1} = \{a^{-1} \mid a \in \Sigma\}$ to denote a disjoint copy of $\Sigma$. Let $\Sigma^{\pm 1} = \Sigma \cup \Sigma^{-1}$. Define $(a^{-1})^{-1} = a$; this defines an involution $^{-1} : \Sigma^{\pm 1} \to \Sigma^{\pm 1}$, which can be extended to $(\Sigma^{\pm 1})^*$ by setting $(a_1 \cdots a_n)^{-1} = a_n^{-1} \cdots a_1^{-1}$. For a word $w \in \Sigma^*$ and $a \in \Sigma$ we denote by $|w|_a$ the number of occurrences of $a$ in $w$. For a subset $\Gamma \subseteq \Sigma$, we denote by $\pi_\Gamma(w)$ the projection of the word $w$ to the alphabet $\Gamma$, i.e., we erase in $w$ all symbols from $\Sigma \setminus \Gamma$.

Let $\mathbb{N}^\Sigma$ be the set of all mappings from $\Sigma$ to $\mathbb{N}$. By fixing an arbitrary linear order on the alphabet $\Sigma$, we may identify a mapping $f \in \mathbb{N}^\Sigma$ with a tuple from $\mathbb{N}^{|\Sigma|}$. For a word $w \in \Sigma^*$, the Parikh image $\Psi(w)$ is defined as the mapping $\Psi(w) : \Sigma \to \mathbb{N}$ such that $[\Psi(w)](a) = |w|_a$ for all $a \in \Sigma$. For a language $L \subseteq \Sigma^*$, the Parikh image is $\Psi(L) = \{\Psi(w) \mid w \in L\}$. For a set $K \subseteq \mathbb{N}^\Sigma$ and $\Gamma \subseteq \Sigma$ let $\overline{\pi}_\Gamma(K) = \{f\restriction_\Gamma \in \mathbb{N}^\Gamma \mid f \in K\}$, where $f\restriction_\Gamma$ denotes the restriction of $f$ to $\Gamma$. We also need a notation for the composition of erasing letters and taking the Parikh image, so, for $L \subseteq \Sigma^*$ and $\Gamma \subseteq \Sigma$, let $\Psi_\Gamma(L) = \overline{\pi}_\Gamma(\Psi(L)) (= \Psi(\pi_\Gamma(L)))$; it may be viewed as a subset of $\mathbb{N}^{|\Gamma|}$. A special case occurs when $\Gamma = \emptyset$. Then either $\Psi_\emptyset(L) = \emptyset$ (if $L = \emptyset$) or $\Psi_\emptyset(L)$ is the singleton set consisting of the unique mapping from $\emptyset$ to $\mathbb{N}$.

A subset $K \subseteq \mathbb{N}^k$ is said to be *linear* if there are $x, x_1, \ldots, x_\ell \in \mathbb{N}^k$ such that $K = \{x + \lambda_1 x_1 + \cdots + \lambda_\ell x_\ell \mid \lambda_1, \ldots, \lambda_\ell \in \mathbb{N}\}$, i.e. $K$ is a translate of a finitely generated submonoid of $\mathbb{N}^k$. A *semilinear* set is a finite union of linear sets.

Let $\mathsf{G} = (N, \Gamma, S, P)$ be a context-free grammar, where $N$ is the set of nonterminals, $\Gamma$ is the terminal alphabet, $S \in N$ is the start nonterminal, and $P \subseteq N \times (N \cup \Gamma)^*$ is the finite set of productions. For $u, v \in (N \cup \Gamma)^*$ we write $u \Rightarrow_{\mathsf{G}} v$ if $v$ can be derived from $u$ by applying a production from $P$. For $A \in N$, we define $L(\mathsf{G}, A) = \{w \in \Gamma^* \mid A \overset{*}{\Rightarrow}_{\mathsf{G}} w\}$ and $L(\mathsf{G}) = L(\mathsf{G}, S)$. Parikh's theorem states that the Parikh image of a context-free language is semilinear [34].

We will allow a more general form of productions in context-free grammars, where the right-hand side of a production is a regular language over the alphabet $N \cup \Gamma$. Such a production $A \to L$ represents the (possibly infinite) set of productions $\{A \to s \mid s \in L\}$. Clearly, such an extended context-free grammar can be transformed into an equivalent context-free grammar with only finitely many productions.

Let $M$ be a monoid. The set $\mathrm{RAT}(M)$ of all *rational subsets* of $M$ is the smallest subset of $2^M$, which contains all finite subsets of $M$, and which is closed under union, product, and Kleene hull (the Kleene hull $L^*$ of a subset $L \subseteq M$ is the submonoid of $M$ generated by $L$). By Kleene's theorem, a subset $L \subseteq \Sigma^*$ is rational if and only if $L$ can be recognized by a finite automaton. If $M$ is generated by the finite set $\Sigma$ and $h : \Sigma^* \to M$ is the corresponding canonical monoid homomorphism, then $L \in \mathrm{RAT}(M)$ if and only if $L = h(K)$ for some $K \in \mathrm{RAT}(\Sigma^*)$. In this case, $L$ can be specified by a finite automaton over the alphabet $\Sigma$. The rational subsets of the free commutative monoid $\mathbb{N}^k$ are exactly the semilinear subsets of $\mathbb{N}^k$ [16].

### 2.2 Groups

Let $G$ be a finitely generated group and let $\Sigma$ be a finite group generating set for $G$. Hence, $\Sigma^{\pm 1}$ is a finite monoid generating set for $G$ and there exists a canonical monoid homomorphism $h : (\Sigma^{\pm 1})^* \to G$. The language

$$\mathrm{WP}_\Sigma(G) = h^{-1}(1)$$

is called the *word problem* of $G$ with respect to $\Sigma$, i.e., $\mathrm{WP}_\Sigma(G)$ consists of all words over the alphabet $\Sigma^{\pm 1}$ which are equal to 1 in the group $G$. It is well known and easy to see that if $\Gamma$ is another finite generating set for $G$, then $\mathrm{WP}_\Sigma(G)$ is decidable if and only if $\mathrm{WP}_\Gamma(G)$ is decidable.

The *submonoid membership problem* for $G$ is the following decision problem:

INPUT: A finite set of words $\Delta \subseteq (\Sigma^{\pm 1})^*$ and a word $w \in (\Sigma^{\pm 1})^*$.
QUESTION: $h(w) \in h(\Delta^*)$?

Note that the subset $h(\Delta^*) \subseteq G$ is the submonoid of $G$ generated by $h(\Delta) \subseteq G$. If we replace in the submonoid membership problem the finitely generated submonoid $h(\Delta^*)$ by the finitely generated subgroup $h((\Delta \cup \Delta^{-1})^*)$, then we obtain the *subgroup membership problem*, which is also known as the *generalized word problem* for $G$. This term is justified, since the word problem is a particular instance, namely with $\Delta = \emptyset$. A generalization of the submonoid membership problem for $G$ is the *rational subset membership problem*:

INPUT: A finite automaton $A$ over the alphabet $\Sigma^{\pm 1}$ and a word $w \in (\Sigma^{\pm 1})^*$.
QUESTION: $h(w) \in h(L(A))$?

Note that $h(w) \in h(L(A))$ if and only if $1 \in h(w^{-1}L(A))$. Since $w^{-1}L(A)$ is again a rational language, the rational subset membership problem for $G$ is recursively equivalent to the decision problem of asking whether $1 \in h(L(A))$ for a given finite automaton $A$ over the alphabet $\Sigma^{\pm 1}$.

In the rational subset (resp. submonoid) membership problem, the rational subset (resp. submonoid) is part of the input. Non-uniform variants of these problems, where the rational subset (resp. submonoid) is fixed, have been studied as well. More generally, we can define for a subset $S \subseteq G$ the *membership problem for $S$ within $G$*:

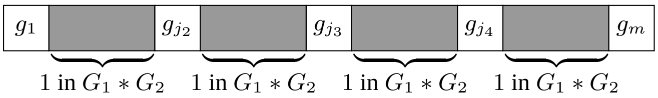


**Fig. 1.** Case (3) in Lemma 1

INPUT: A word $w \in (\Sigma^{\pm 1})^*$.
QUESTION: $h(w) \in S$?

It should be noted that for all the computational problems introduced above the decidability/complexity is independent of the chosen generating set for $G$.

The *free group* $F(\Sigma)$ generated by $\Sigma$ can be defined as the quotient monoid

$$F(\Sigma) = (\Sigma^{\pm 1})^* / \{aa^{-1} = \varepsilon \mid a \in \Sigma^{\pm 1}\}.$$

As usual, the *free product* of two groups $G_1$ and $G_2$ is denoted by $G_1 * G_2$. We will always assume that $G_1 \cap G_2 = \emptyset$. An *alternating word* in $G_1 * G_2$ is a sequence $g_1 g_2 \cdots g_m$ with $g_i \in G_1 \cup G_2$ and $g_i \in G_1 \Leftrightarrow g_{i+1} \in G_2$. Its length is $m$. The alternating word $g_1 g_2 \cdots g_m$ is *irreducible* if $g_i \neq 1$ for every $1 \leq i \leq m$. Every element of $G_1 * G_2$ can be written uniquely as an alternating irreducible word. We will need the following simple fact about free products:

**Lemma 1.** *Let $g_1 g_2 \cdots g_m$ be an alternating word in $G_1 * G_2$. If $g_1 g_2 \cdots g_m = 1$ in $G_1 * G_2$, then one of the following three cases holds:*

*(1) $m \leq 1$*
*(2) there exists $1 \leq i < m$ such that $g_1 g_2 \cdots g_i = g_{i+1} \cdots g_m = 1$ in $G_1 * G_2$*
*(3) there exist $i \in \{1, 2\}$, $k \geq 2$, and $1 = j_1 < j_2 < \cdots < j_k = m$ such that $g_{j_1}, g_{j_2}, \ldots, g_{j_k} \in G_i$, $g_{j_1} g_{j_2} \cdots g_{j_k} = 1$ in $G_i$, and $g_{j_\ell + 1} g_{j_\ell + 2} \cdots g_{j_{\ell+1} - 1} = 1$ in $G_1 * G_2$ for all $1 \leq \ell < k$.*

*Proof.* Case (3) from the Lemma is visualized in Figure 1 for $k = 5$. Shaded areas represent alternating sequences, which are equal to 1 in $G_1 * G_2$. The non-shaded blocks are either all from $G_1$ or from $G_2$, and their product equals 1 in $G_1$ or $G_2$, respectively.

We prove the lemma by induction over $m$, the case $m \leq 1$ being trivial. So assume that $m \geq 2$. Since $g_1 g_2 \cdots g_m = 1$ in $G_1 * G_2$, there must exist $1 \leq j \leq m$ with $g_j = 1$. If $j = 1$ or $j = m$, then we obtain case (2) from the lemma. Hence, we may assume that $m \geq 3$ and that $2 \leq j \leq m - 1$. It follows

$$g_1 \cdots g_{j-2} (g_{j-1} g_{j+1}) g_{j+2} \cdots g_m = 1$$

in $G_1 * G_2$. Since the alternating word $g_1 \cdots g_{j-2} (g_{j-1} g_{j+1}) g_{j+2} \cdots g_m$ has length $m - 2$, we can apply the induction hypothesis to it. If $m - 2 = 1$, i.e., $m = 3$, then we obtain case (3) from the lemma (with $k = 2$, $j_1 = 2$, and $j_2 = 3$). If a non-empty and proper prefix of $g_1 \cdots g_{j-2} (g_{j-1} g_{j+1}) g_{j+2} \cdots g_m$ equals 1 in the group $G_1 * G_2$, then the same is true for $g_1 g_2 \cdots g_m$. Finally, if case (3) from the lemma applies to the alternating word $g_1 \cdots g_{j-2} (g_{j-1} g_{j+1}) g_{j+2} \cdots g_m$, then again the same is true for $g_1 g_2 \cdots g_m$. $\square$

Notice that (3) in Lemma 1 can only occur when $m$ is odd.

We will also consider fundamental groups of finite graphs of groups, which is a group theoretic construction generalizing free products, free products with amalgamation and HNN-extensions, see e.g. [36]. We omit the quite technical definition. In order to deal with the rational subset membership problem for graph groups, free products suffice.

### 2.3 Trace monoids and graph groups

In the following we introduce some notions from trace theory, see [11, 14] for more details. An *independence alphabet* is just a finite undirected graph $(\Sigma, I)$ without loops. Hence, $I \subseteq \Sigma \times \Sigma$ is an irreflexive and symmetric relation. The *trace monoid* $\mathbb{M}(\Sigma, I)$ is defined as the quotient

$$\mathbb{M}(\Sigma, I) = \Sigma^*/\{ab = ba \mid (a, b) \in I\}.$$

Elements of $\mathbb{M}(\Sigma, I)$ are called *traces*.

Traces can be represented conveniently by *dependence graphs*, which are node-labelled directed acyclic graphs. Let $u = a_1 \cdots a_n$ be a word, where $a_i \in \Sigma$. The vertex set of the dependence graph of $u$ is $\{1, \ldots, n\}$ and vertex $i$ is labelled with $a_i \in \Sigma$. There is an edge from vertex $i$ to $j$ if and only if $i < j$ and $(a_i, a_j) \notin I$. Then, two words define the same trace in $\mathbb{M}(\Sigma, I)$ if and only if their dependence graphs are isomorphic. The set of minimal (resp. maximal) elements of a trace $t \in \mathbb{M}(\Sigma, I)$ is $\min(t) = \{a \in \Sigma \mid \exists u \in \mathbb{M}(\Sigma, I) : t = au\}$ (resp. $\max(t) = \{a \in \Sigma \mid \exists u \in \mathbb{M}(\Sigma, I) : t = ua\}$). A *trace rewriting system* $R$ over $\mathbb{M}(\Sigma, I)$ is just a finite subset of $\mathbb{M}(\Sigma, I) \times \mathbb{M}(\Sigma, I)$ [11]. We can define the *one-step rewrite relation* $\to_R \subseteq \mathbb{M}(\Sigma, I) \times \mathbb{M}(\Sigma, I)$ by: $x \to_R y$ if and only if there are $u, v \in \mathbb{M}(\Sigma, I)$ and $(\ell, r) \in R$ such that $x = u\ell v$ and $y = urv$. The notion of a *confluent* and *terminating* trace rewriting system is defined as for other types of rewriting systems [5]. A trace $t$ is *irreducible* with respect to $R$ if there does not exist a trace $u$ with $t \to_R u$. If $R$ is terminating and confluent, then for every trace $t$, there exists a unique *normal form* $\mathrm{NF}_R(t)$ such that $t \xrightarrow{*}_R \mathrm{NF}_R(t)$ and $\mathrm{NF}_R(t)$ is irreducible with respect to $R$.

The *graph group* $\mathbb{G}(\Sigma, I)$ is defined as the quotient

$$\mathbb{G}(\Sigma, I) = F(\Sigma)/\{ab = ba \mid (a, b) \in I\}.$$

If $(\Sigma, I)$ is the empty graph, i.e., $\Sigma = \emptyset$, then we set $\mathbb{M}(\Sigma, I) = \mathbb{G}(\Sigma, I) = 1$ (the trivial group). Note that $(a, b) \in I$ implies $a^{-1}b = ba^{-1}$ in $\mathbb{G}(\Sigma, I)$. Thus, the graph group $\mathbb{G}(\Sigma, I)$ can be also defined as the quotient

$$\mathbb{G}(\Sigma, I) = \mathbb{M}(\Sigma^{\pm 1}, I)/\{aa^{-1} = \varepsilon \mid a \in \Sigma^{\pm 1}\}.$$

Here, we implicitly extend $I \subseteq \Sigma \times \Sigma$ to $I \subseteq \Sigma^{\pm 1} \times \Sigma^{\pm 1}$ by setting $(a^\alpha, b^\beta) \in I$ if and only if $(a, b) \in I$ for $a, b \in \Sigma$ and $\alpha, \beta \in \{1, -1\}$. Note that $\mathbb{M}(\Sigma, I)$ is a rational subset of $\mathbb{G}(\Sigma, I)$.

Define a trace rewriting system $R$ over $\mathbb{M}(\Sigma^{\pm 1}, I)$ as follows:

$$R = \{(aa^{-1}, \varepsilon) \mid a \in \Sigma^{\pm 1}\}. \tag{1}$$

One can show that $R$ is terminating and confluent and that for all $u, v \in \mathbb{M}(\Sigma^{\pm 1}, I)$: $u = v$ in $\mathbb{G}(\Sigma, I)$ if and only if $\mathrm{NF}_R(u) = \mathrm{NF}_R(v)$ [12]. This leads to a linear time solution for the word problem of $\mathbb{G}(\Sigma, I)$ [12, 38].

If the graph $(\Sigma, I)$ is the disjoint union of two graphs $(\Sigma_1, I_1)$ and $(\Sigma_2, I_2)$, then $\mathbb{G}(\Sigma, I) = \mathbb{G}(\Sigma_1, I_1) * \mathbb{G}(\Sigma_2, I_2)$. If $(\Sigma, I)$ is obtained from $(\Sigma_1, I_1)$ and $(\Sigma_2, I_2)$ by connecting each element of $\Sigma_1$ to each element of $\Sigma_2$, then $\mathbb{G}(\Sigma, I) = \mathbb{G}(\Sigma_1, I_1) \times \mathbb{G}(\Sigma_2, I_2)$. Graph groups were studied e.g. in [15]; they are also known as *free partially commutative groups* [12, 38], *right-angled Artin groups* [7, 9], and *semifree groups* [2].

A transitive forest is an independence alphabet $(\Sigma, I)$ such that there exists a forest $F$ of rooted trees (i.e., a disjoint union of rooted trees) with node set $\Sigma$ and such that for all $a, b \in \Sigma$ with $a \neq b$: $(a, b) \in I$ if and only if $a$ and $b$ are comparable in $F$ (i.e., either $a$ is a proper descendant of $b$ or $b$ is a proper descendant of $a$). It can be shown that $(\Sigma, I)$ is a transitive forest if and only if $(\Sigma, I)$ does not contain an induced subgraph, which is a cycle on 4 nodes (also called C4) or a simple path on 4 nodes (also called P4) [37]. The next lemma follows easily by induction. We give a sketch of the proof.

**Lemma 2.** *The class $C$ of all groups, which are of the form $\mathbb{G}(\Sigma, I)$ for a transitive forest $(\Sigma, I)$, is the smallest class such that:*

*(1)* $1 \in C$
*(2) if $G_1, G_2 \in C$, then also $G_1 * G_2 \in C$*
*(3) if $G \in C$ then $G \times \mathbb{Z} \in C$*

*Proof.* First we verify that graphs groups associated to transitive forests satisfy (1)-(3). Case (1) is the empty graph. It is immediate that transitive forests are closed under disjoint union, which implies (2). If $F$ is a forest of rooted trees, then one can obtain a rooted tree by adding a new root whose children are the roots of the trees from $F$. On the group level this corresponds to (3).

For the converse, we proceed by induction on the number of vertices. If the forest $(\Sigma, I)$ consists of more than one rooted tree, then $\mathbb{G}(\Sigma, I)$ is the free product of the graph groups associated to the various rooted trees in $(\Sigma, I)$, all of which have a smaller number of vertices. If there is a single tree, then in $(\Sigma, I)$ the root is connected to every other vertex. Thus $\mathbb{G}(\Sigma, I) = G \times \mathbb{Z}$ where $G$ is the graph group corresponding to the transitive forest obtained by removing the vertex corresponding to the root and making its children the roots of the trees in the forest so obtained. □

Of course, a similar statement is true for trace monoids of the form $\mathbb{M}(\Sigma, I)$ with $(\Sigma, I)$ a transitive forest; one just has to replace in (3) the group $\mathbb{Z}$ by the monoid $\mathbb{N}$.

## 3 The rational subset membership problem

Let $\mathcal{C}$ be the smallest class of groups such that:

- the trivial group 1 belongs to $\mathcal{C}$
- if $G \in \mathcal{C}$ and $H \leq G$ is finitely generated, then also $H \in \mathcal{C}$

– if $G \in \mathcal{C}$ and $G \leq H$ such that $G$ has finite index in $H$ (i.e., $H$ is a finite extension of $G$), then also $H \in \mathcal{C}$
– if $G \in \mathcal{C}$, then also $G \times \mathbb{Z} \in \mathcal{C}$
– if $\mathbb{A}$ is a finite graph of groups [36] whose edge groups are finite and whose vertex groups belong to $\mathcal{C}$, then the fundamental group of $\mathbb{A}$ belongs to $\mathcal{C}$ (in particular, the class $\mathcal{C}$ is closed under free products).

This last property is equivalent to saying that $\mathcal{C}$ is closed under taking amalgamated products over finite groups and HNN-extensions with finite associated subgroups [36]. The main result in this section is:

**Theorem 1.** *For every group $G \in \mathcal{C}$, the rational subset membership problem is decidable.*

It is well known that decidability of the rational subset membership problem is preserved under taking subgroups and finite extensions [21]. Moreover, the decidability of the rational subset membership problem is preserved by graph of group constructions with finite edge groups [25]. Hence, in order to prove Theorem 1, it would suffice to show that the decidability of the rational subset membership problem is preserved under direct products by $\mathbb{Z}$. But currently we can neither prove nor disprove this. Hence, we shall follow another strategy. We will introduce a property of groups that implies the decidability of the rational subset membership problem, and which has the desired closure properties.

Let $\mathcal{L}$ be a class of formal languages closed under inverse homomorphism. A finitely generated group $G$ is said to be an *$\mathcal{L}$-group* if $\mathrm{WP}_{\Sigma}(G)$ belongs to $\mathcal{L}$ for some finite generating set $\Sigma$. This notion is independent of the choice of generating set [18, 22, 25].

A language $L_0 \subseteq \Sigma^*$ belongs to the class *RID* (rational intersection decidable) if there is an algorithm that, given a finite automaton over $\Sigma$ recognizing a rational language $L$, can determine whether $L_0 \cap L \neq \emptyset$. It was shown in [25] that the class RID is closed under inverse homomorphism and that a group $G$ has a decidable rational subset membership problem if and only if it is an RID-group. This follows from the fact that if $L$ is a rational subset of a group $G$, then $g \in L$ if and only if $1 \in g^{-1}L$ and that $g^{-1}L$ is again a rational subset.

Let $K \subseteq \Theta^*$ be a language over an alphabet $\Theta$. Then $K$ belongs to the class SLI (semilinear intersection) if, for every finite alphabet $\Gamma$ (disjoint from $\Theta$) and every rational language $L \subseteq (\Theta \cup \Gamma)^*$, the set

$$\Psi_{\Gamma}(\{w \in L \mid \pi_{\Theta}(w) \in K\}) = \Psi_{\Gamma}(L \cap \pi_{\Theta}^{-1}(K)) \tag{2}$$

is semilinear, and the tuples in a semilinear representation of this set can be effectively computed from $\Gamma$ and a finite automaton for $L$. This latter effectiveness statement will be always satisfied throughout the paper, and we shall not explicitly check it. In words, the set (2) is obtained by first taking those words from $L$ that project into $K$ when $\Gamma$-letters are erased, and then erasing the $\Gamma$-letters, followed by taking the Parikh image.

In a moment, we shall see that the class SLI is closed under inverse homomorphism, hence the class of SLI-groups is well defined. In fact, we show more generally that the class SLI is closed under inverse images by finite state subsequential functions.

This will imply, moreover, that the class of SLI-groups is closed under taking finite extensions [18, 22, 25].

Recall that a finite state automaton over $\Sigma \times \Omega^*$ is called a *subsequential transducer*, if the subset recognized by it is the graph of a partial function $f : \Sigma^* \to \Omega^*$, called a *finite state subsequential function*.

**Lemma 3.** *Let $K \subseteq \Theta^*$ belong to SLI and let $f : \Sigma^* \to \Theta^*$ be a finite state subsequential function. Then $f^{-1}(K)$ belongs to SLI. In particular, the class of SLI-groups is well defined and is closed under taking finite extensions.*

*Proof.* Let $\Gamma$ be an alphabet disjoint from $\Sigma$ and let $L$ be a rational subset of $(\Gamma \cup \Sigma)^*$. Let $A$ be a subsequential transducer computing $f : \Sigma^* \to \Omega^*$. Define a finite state subsequential function $F : (\Gamma \cup \Sigma)^* \to (\Gamma \cup \Theta)^*$ by adding to each state of $A$ a loop with label $(a, a)$ for each $a \in \Gamma$. Call the resulting transducer $A'$.

The following two observations are immediate from the fact that the only transitions of $A'$ involving letters from $\Gamma$ are loops with labels of the form $(a, a)$:

(a) $\Psi_\Gamma F$ coincides with $\Psi_\Gamma$ on the domain of $F$ (we read the composition of functions from right to left, i.e., in $\Psi_\Gamma F$ we first apply $F$, followed by $\Psi_\Gamma$)
(b) $\pi_\Theta F = f \pi_\Sigma$.

We now claim that the following equality holds:

$$F(L \cap \pi_\Sigma^{-1}(f^{-1}(K))) = F(L) \cap \pi_\Theta^{-1}(K). \quad (3)$$

First note that $L \cap \pi_\Sigma^{-1}(f^{-1}(K)) = L \cap F^{-1}(\pi_\Theta^{-1}(K))$ by (b). So if $w$ belongs to the left hand side of (3), then $w = F(u)$ with $u \in L \cap F^{-1}(\pi_\Theta^{-1}(K))$. Thus $w \in F(L) \cap \pi_\Theta^{-1}(K)$. Conversely, if $u \in F(L) \cap \pi_\Theta^{-1}(K)$, then there exists $w \in L$ such that $F(w) = u$. But then $w \in L \cap F^{-1}(\pi_\Theta^{-1}(K)) = L \cap \pi_\Sigma^{-1}(f^{-1}(K))$ and so $u$ belongs to the left hand side of (3).

Now, since $L \cap \pi_\Sigma^{-1}(f^{-1}(K)) = L \cap F^{-1}(\pi_\Theta^{-1}(K))$ is contained in the domain of $F$, we may conclude from (a) and (3) that

$$\Psi_\Gamma(L \cap \pi_\Sigma^{-1}(f^{-1}(K))) = \Psi_\Gamma F(L \cap \pi_\Sigma^{-1}(f^{-1}(K))) = \Psi_\Gamma(F(L) \cap \pi_\Theta^{-1}(K)). \quad (4)$$

But $F(L)$ is rational since the class of rational languages is closed under images via finite state subsequential functions [3]. Therefore, since $K$ belongs to SLI, we may deduce that $\Psi_\Gamma(F(L) \cap \pi_\Theta^{-1}(K))$ is semilinear. This completes the proof of the first statement from the theorem in light on (4).

Since a homomorphism is a finite state subsequential function, the language class SLI is closed under inverse homomorphism. Hence, the class of SLI-groups is well defined. Finally, let us assume that $G$ is an SLI-group and that $G$ is a finite index subgroup of $H$. Let $\Sigma$ (resp. $\Delta$) be a finite generating set for $G$ (resp. $H$). Then in [25, Lemma 3.3] it is shown that there exists a finite state subsequential function $f : \Delta^* \to \Sigma^*$ such that $\mathrm{WP}_\Delta(H) = f^{-1}(\mathrm{WP}_\Sigma(G))$. Hence, $H$ is an SLI-group. □

Let us quickly dispense with the decidability of the rational subset membership problem for SLI-groups.

**Lemma 4.** *The class of languages SLI is contained in the class of languages RID. In particular, every SLI-group has a decidable rational subset membership problem.*

*Proof.* Let $K \subseteq \Theta^*$ belong to SLI. Let $A$ be a finite automaton over the alphabet $\Theta$. We have to decide whether $L(A) \cap K \neq \emptyset$. Since $K$ belongs to SLI, the set

$$\Psi_\emptyset(\{w \in L(A) \mid \pi_\Theta(w) \in K\}) = \Psi_\emptyset(L(A) \cap K)$$

is effectively semilinear and so has a decidable membership problem (c.f. [25]). As mentioned earlier, $\Psi_\emptyset(L(A) \cap K)$ consists of the unique function $\emptyset \to \mathbb{N}$ if $L(A) \cap K$ is non-empty and is empty otherwise. Thus we can test emptiness for $L(A) \cap K$. □

Having already taken care of finite extensions by Lemma 3, let's turn to finitely generated subgroups. We show that the language class SLI is closed under intersection with rational subsets. This guarantees that the class of SLI-groups is closed under taking finitely generated subgroups [22].

**Lemma 5.** *Let $K \subseteq \Theta^*$ belong to SLI and let $R \subseteq \Theta^*$ be rational. Then $R \cap K$ belongs to SLI. In particular, every finitely generated subgroup of an SLI-group is an SLI-group.*

*Proof.* Let $L \subseteq (\Gamma \cup \Theta)^*$ be rational. We have $L \cap \pi_\Theta^{-1}(R \cap K) = L \cap \pi_\Theta^{-1}(R) \cap \pi_\Theta^{-1}(K)$. But rational languages are closed under inverse homomorphism and intersection, so $\Psi_\Gamma(L \cap \pi_\Theta^{-1}(R) \cap \pi_\Theta^{-1}(K))$ is semilinear as $K$ belongs to SLI. This establishes the lemma. □

Next, we show that the class of SLI-groups is closed under direct products with $\mathbb{Z}$:

**Lemma 6.** *If $G$ is an SLI-group, then $G \times \mathbb{Z}$ is also an SLI-group.*

*Proof.* Let $\Sigma$ be a finite generating set for $G$. Choose a generator $a \notin \Sigma$ of $\mathbb{Z}$. Then $G \times \mathbb{Z}$ is generated by $\Sigma \cup \{a\}$. Let $\Gamma$ be a finite alphabet ($\Gamma \cap (\Sigma^{\pm 1} \cup \{a, a^{-1}\}) = \emptyset$) and let $L$ be a rational subset of $(\Sigma^{\pm 1} \cup \{a, a^{-1}\} \cup \Gamma)^*$. We have

$$\begin{aligned}\Psi_\Gamma\Big(\{w \in L \mid \pi_{\Sigma^{\pm 1} \cup \{a,a^{-1}\}}(w) \in \mathrm{WP}_{\Sigma \cup \{a\}}(G \times \mathbb{Z})\}\Big) = \\ \overline{\pi}_\Gamma\Big(\ \Psi_{\Gamma \cup \{a,a^{-1}\}}(\{w \in L \mid \pi_{\Sigma^{\pm 1}}(w) \in \mathrm{WP}_\Sigma(G)\}) \cap \\ \{f \in \mathbb{N}^{\Gamma \cup \{a,a^{-1}\}} \mid f(a) = f(a^{-1})\}\Big).\end{aligned}$$

This set is semilinear, since $\{f \in \mathbb{N}^{\Gamma \cup \{a,a^{-1}\}} \mid f(a) = f(a^{-1})\}$ is semilinear and semilinear sets are closed under intersection and projection [19]. □

By Lemma 3–6, Theorem 1 would be established, if we could prove the closure of $\mathcal{C}$ under graph of groups constructions with finite edge groups. Unfortunately we are only able to prove this closure under the restriction that every vertex group of the graph of groups is residually finite (which is the case for groups in $\mathcal{C}$). In general we can just prove closure under free product. This, in fact, constitutes the most difficult part of the proof of Theorem 1.

**Lemma 7.** *If $G_1$ and $G_2$ are SLI-groups, then $G_1 * G_2$ is also an SLI-group.*

*Proof.* Assume that $\Sigma_i$ is a finite generating set for $G_i$. Thus, $\Sigma = \Sigma_1 \cup \Sigma_2$ is a generating set for the free product $G_1 * G_2$. Let $\Gamma$ be a finite alphabet ($\Gamma \cap \Sigma^{\pm 1} = \emptyset$) and let $\Theta = \Sigma^{\pm 1} \cup \Gamma$. Let $L \subseteq \Theta^*$ be rational and let $A = (Q, \Theta, \delta, q_0, F)$ be a finite automaton with $L = L(A)$, where $Q$ is the set of states, $\delta \subseteq Q \times \Theta \times Q$ is the transition relation, $q_0 \in Q$ is the initial state, and $F \subseteq Q$ is the set of final states. For $p, q \in Q$ and $w \in \Theta^*$ we write $p \xrightarrow{w}_A q$ if there exists a path in $A$ from $p$ to $q$, labelled by the word $w$.

For every pair of states $(p, q) \in Q \times Q$ let us define the language

$$L[p,q] \subseteq (\Sigma_1^{\pm 1} \cup \Gamma \cup (Q \times Q))^* \cup (\Sigma_2^{\pm 1} \cup \Gamma \cup (Q \times Q))^* \subseteq (\Theta \cup (Q \times Q))^*$$

as follows:

$$\begin{aligned} L[p,q] = \bigcup_{i \in \{1,2\}} \{ & w_0(p_1,q_1)w_1(p_2,q_2)\cdots w_{k-1}(p_k,q_k)w_k \mid \\ & k \geq 1 \ \wedge\ (p_1,q_1),\ldots,(p_k,q_k) \in Q \times Q \ \wedge \\ & w_0,\ldots,w_k \in (\Sigma_i^{\pm 1} \cup \Gamma)^* \ \wedge\ \pi_{\Sigma_i^{\pm 1}}(w_0 \cdots w_k) \in \mathrm{WP}_{\Sigma_i}(G_i) \ \wedge \\ & p \xrightarrow{w_0}_A p_1 \ \wedge\ q_1 \xrightarrow{w_1}_A p_2 \ \wedge \cdots \wedge\ q_{k-1} \xrightarrow{w_{k-1}}_A p_k \ \wedge\ q_k \xrightarrow{w_k}_A q \} \end{aligned}$$

Since the language

$$\begin{aligned} \{ & w_0(p_1,q_1)w_1(p_2,q_2)\cdots w_{k-1}(p_k,q_k)w_k \mid \\ & k \geq 1 \ \wedge\ (p_1,q_1),\ldots,(p_k,q_k) \in Q \times Q \ \wedge\ w_0,\ldots,w_k \in (\Sigma_i^{\pm 1} \cup \Gamma)^* \ \wedge \\ & p \xrightarrow{w_0}_A p_1 \ \wedge\ q_1 \xrightarrow{w_1}_A p_2 \ \wedge \cdots \wedge\ q_{k-1} \xrightarrow{w_{k-1}}_A p_k \ \wedge\ q_k \xrightarrow{w_k}_A q \} \end{aligned}$$

is a rational language over the alphabet $\Sigma_i^{\pm 1} \cup \Gamma \cup (Q \times Q)$ for $i \in \{1, 2\}$ and $G_i$ is an SLI-group, it follows that the Parikh image $\Psi_{\Gamma \cup (Q \times Q)}(L[p,q]) \subseteq \mathbb{N}^{\Gamma \cup (Q \times Q)}$ is semilinear. Let $K[p,q] \subseteq (\Gamma \cup (Q \times Q))^*$ be some rational language such that

$$\Psi(K[p,q]) = \Psi_{\Gamma \cup (Q \times Q)}(L[p,q]). \tag{5}$$

Next, we define a context-free grammar $\mathsf{G} = (N, \Gamma, S, P)$ as follows:

- the set of nonterminals is $N = \{S\} \uplus (Q \times Q)$.
- $S$ is the start nonterminal.
- $P$ consists of the following productions:

$$\begin{aligned} S &\to (q_0, q_f) && \text{for all } q_f \in F \\ (p,q) &\to K[p,q] && \text{for all } p, q \in Q \\ (q,q) &\to \varepsilon && \text{for all } q \in Q \end{aligned}$$

By Parikh's theorem, the Parikh image $\Psi(L(\mathsf{G})) \subseteq \mathbb{N}^{\Gamma}$ is semilinear. Thus, the following claim proves the lemma:

*Claim 1.* $\Psi(L(\mathsf{G})) = \Psi_\Gamma(\{w \in L(A) \mid \pi_{\Sigma^{\pm 1}}(w) \in \mathsf{WP}_\Sigma(G_1 * G_2)\})$

*Proof of Claim 1.* We prove the following more general identity for all $(p, q) \in Q \times Q$:

$$\Psi(L(\mathsf{G}, (p, q))) = \Psi_\Gamma(\{w \in \Theta^* \mid p \xrightarrow{w}_A q \ \wedge \ \pi_{\Sigma^{\pm 1}}(w) \in \mathsf{WP}_\Sigma(G_1 * G_2)\})$$

For the inclusion from left to right assume that $(p, q) \stackrel{*}{\Rightarrow}_{\mathsf{G}} u \in \Gamma^*$. We show by induction on the length of the $\mathsf{G}$-derivation $(p, q) \stackrel{*}{\Rightarrow}_{\mathsf{G}} u$ that there exists a word $w \in \Theta^*$ such that $p \xrightarrow{w}_A q$, $\pi_{\Sigma^{\pm 1}}(w) \in \mathsf{WP}_\Sigma(G_1 * G_2)$, and $\Psi(u) = \Psi_\Gamma(w)$.

*Case 1.* $p = q$ and $u = \varepsilon$: We can choose $w = \varepsilon$.

*Case 2.* $(p, q) \Rightarrow_{\mathsf{G}} u' \stackrel{*}{\Rightarrow}_{\mathsf{G}} u$ for some $u' \in K[p, q]$. By (5), there exists a word $v \in L[p, q]$ such that $\Psi(u') = \Psi_{\Gamma \cup (Q \times Q)}(v)$. Since $v \in L[p, q]$, there exist $k \geq 1$, $(p_1, q_1), \ldots, (p_k, q_k) \in Q \times Q$, $i \in \{1, 2\}$, and $v_0, \ldots, v_k \in (\Sigma_i^{\pm 1} \cup \Gamma)^*$ such that

- $p \xrightarrow{v_0}_A p_1,\ q_1 \xrightarrow{v_1}_A p_2, \ldots, q_{k-1} \xrightarrow{v_{k-1}}_A p_k,\ q_k \xrightarrow{v_k}_A q$,
- $v = v_0(p_1, q_1)v_1(p_2, q_2) \cdots v_{k-1}(p_k, q_k)v_k$, and
- $\pi_{\Sigma_i^{\pm 1}}(v_0 \cdots v_k) \in \mathsf{WP}_{\Sigma_i}(G_i)$.

Since $u' \stackrel{*}{\Rightarrow}_{\mathsf{G}} u \in \Gamma^*$ and $\Psi(u') = \Psi_{\Gamma \cup (Q \times Q)}(v)$, there must exist $u_1, \ldots, u_k \in \Gamma^*$ such that

$$(p_i, q_i) \stackrel{*}{\Rightarrow}_{\mathsf{G}} u_i \quad \text{and} \quad \Psi(u) = \Psi_\Gamma(v_0) + \cdots + \Psi_\Gamma(v_k) + \Psi(u_1) + \cdots + \Psi(u_k)$$

for all $1 \leq i \leq k$. By induction, we obtain words $w_1, \ldots, w_k \in \Theta^*$ such that for all $1 \leq i \leq k$:

- $p_i \xrightarrow{w_i}_A q_i$
- $\pi_{\Sigma^{\pm 1}}(w_i) \in \mathsf{WP}_\Sigma(G_1 * G_2)$, and
- $\Psi(u_i) = \Psi_\Gamma(w_i)$.

Let us set $w = v_0 w_1 v_1 \cdots w_k v_k \in \Theta^*$. We have:

- $p \xrightarrow{v_0}_A p_1 \xrightarrow{w_1}_A q_1 \xrightarrow{v_1}_A p_2 \cdots p_k \xrightarrow{w_k}_A q_k \xrightarrow{v_k}_A q$, i.e., $p \xrightarrow{w}_A q$,
- $\pi_{\Sigma^{\pm 1}}(w) \in \mathsf{WP}_\Sigma(G_1 * G_2)$, and
- $\Psi(u) = \Psi_\Gamma(v_0) + \cdots + \Psi_\Gamma(v_k) + \Psi(u_1) + \cdots + \Psi(u_k) = \Psi_\Gamma(v_0) + \cdots + \Psi_\Gamma(v_k) + \Psi_\Gamma(w_1) + \cdots + \Psi_\Gamma(w_k) = \Psi_\Gamma(w)$.

This concludes the proof of the inclusion

$$\Psi(L(\mathsf{G}, (p, q))) \subseteq \Psi_\Gamma(\{w \in \Theta^* \mid p \xrightarrow{w}_A q \ \wedge \ \pi_{\Sigma^{\pm 1}}(w) \in \mathsf{WP}_\Sigma(G_1 * G_2)\}).$$

For the other inclusion, assume that

$$p \xrightarrow{w}_A q \quad \text{and} \quad \pi_{\Sigma^{\pm 1}}(w) \in \mathsf{WP}_\Sigma(G_1 * G_2)$$

for a word $w \in \Theta^*$. By induction over the length of the word $w$ we show that $\Psi_\Gamma(w) \in \Psi(L(\mathsf{G}, (p, q)))$.

We will make a case distinction according to the three cases in Lemma 1. Note that we either have $w \in \Gamma^*$ or the word $w \in \Theta^*$ can be (not necessarily uniquely) written as $w = w_1 \cdots w_n$ with $n \geq 1$ such that $w_i \in ((\Gamma \cup \Sigma_1^{\pm 1})^* \cup (\Gamma \cup \Sigma_2^{\pm 1})^*) \setminus \Gamma^*$ and $w_i \in (\Gamma \cup \Sigma_1^{\pm 1})^* \Leftrightarrow w_{i+1} \in (\Gamma \cup \Sigma_2^{\pm 1})^*$.

*Case 1.* $w \in (\Gamma \cup \Sigma_1^{\pm 1})^*$ (the case $w \in (\Gamma \cup \Sigma_2^{\pm 1})^*$ is analogous): Then $\pi_{\Sigma_1^{\pm 1}}(w) \in \mathrm{WP}_{\Sigma_1}(G_1)$. Together with $p \xrightarrow{w}_A q$, we obtain $w(q,q) \in L[p,q]$. Since $(p,q) \to K[p,q]$ and $(q,q) \to \varepsilon$ are productions of $\mathsf{G}$, there exists a word $u \in \Gamma^*$ such that $(p,q) \xRightarrow{*}_{\mathsf{G}} u$ and $\Psi(u) = \Psi_\Gamma(w)$, i.e., $\Psi_\Gamma(w) \in \Psi(L(\mathsf{G},(p,q)))$.

*Case 2.* $w = w_1 w_2$ with $w_1 \neq \varepsilon \neq w_2$ and $\pi_{\Sigma^{\pm 1}}(w_1), \pi_{\Sigma^{\pm 1}}(w_2) \in \mathrm{WP}_\Sigma(G_1 * G_2)$. Then there exists a state $r \in Q$ such that

$$p \xrightarrow{w_1}_A r \xrightarrow{w_2}_A q.$$

By induction, we obtain

$$\Psi_\Gamma(w_1) \in \Psi(L(\mathsf{G},(p,r))) \text{ and}$$
$$\Psi_\Gamma(w_2) \in \Psi(L(\mathsf{G},(r,q))).$$

Hence, we get

$$\begin{aligned}\Psi_\Gamma(w) &= \Psi_\Gamma(w_1) + \Psi_\Gamma(w_2)\\ &\in \Psi(L(\mathsf{G},(p,r))) + \Psi(L(\mathsf{G},(r,q)))\\ &\subseteq \Psi(L(\mathsf{G},(p,q))),\end{aligned}$$

where the last inclusion holds, since $(p,r)(r,q) \in L[p,q]$, and so either $(p,q) \to (p,r)(r,q)$ or $(p,q) \to (r,q)(p,r)$ is a production of $\mathsf{G}$.

*Case 3.* $w = v_0 w_1 v_1 \cdots w_k v_k$ such that $k \geq 1$,

- $\pi_{\Sigma^{\pm 1}}(w_i) \in \mathrm{WP}_\Sigma(G_1 * G_2)$ for all $i \in \{1, \ldots, k\}$, and
- for some $i \in \{1,2\}$: $v_0, \ldots, v_k \in (\Gamma \cup \Sigma_i^{\pm 1})^* \setminus \Gamma^*$ and $\pi_{\Sigma_i^{\pm 1}}(v_0 \cdots v_k) \in \mathrm{WP}_{\Sigma_i}(G_i)$.

There exist states $p_1, q_1, \ldots, p_k, q_k \in Q$ such that

$$p \xrightarrow{v_0}_A p_1 \xrightarrow{w_1}_A q_1 \xrightarrow{v_1}_A p_2 \cdots p_k \xrightarrow{w_k}_A q_k \xrightarrow{v_k}_A q.$$

By induction, we obtain

$$\Psi_\Gamma(w_i) \in \Psi(L(\mathsf{G},(p_i,q_i))) \tag{6}$$

for all $1 \leq i \leq k$. Moreover, from the definition of the language $L[p,q]$ we obtain

$$v = v_0(p_1,q_1)v_1(p_2,q_2) \cdots v_{k-1}(p_k,q_k)v_k \in L[p,q].$$

Hence, there is a word $u' \in K[p,q]$ such that $\Psi(u') = \Psi_{\Gamma \cup (Q \times Q)}(v)$ and $(p,q) \to u'$ is a production of $G$. With (6) we obtain

$$(p,q) \Rightarrow_{\mathsf{G}} u' \xRightarrow{*}_{\mathsf{G}} u$$

for a word $u \in \Gamma^*$ such that

$$\Psi(u) = \Psi_\Gamma(v_0) + \cdots + \Psi_\Gamma(v_k) + \Psi_\Gamma(w_1) + \cdots + \Psi_\Gamma(w_k) = \Psi_\Gamma(w),$$

i.e., $\Psi_\Gamma(w) \in \Psi(L(\mathsf{G},(p,q)))$. This concludes the proof of Claim 1. $\square$

If we were to weaken the definition of the class $\mathcal{C}$ by only requiring closure under free products instead of closure under finite graphs of groups with finite edge groups, then Lemma 4–7 would already imply Theorem 1. In fact, this weaker result suffices in order to deal with graph groups, and readers only interested in graph groups can skip the following considerations concerning graphs of groups.

To obtain the more general closure result for the class $\mathcal{C}$ concerning graph of group constructions, we reduce to the case of free products. Recall that a group $G$ is *residually finite* if, for each $g \in G \setminus \{1\}$, there is a finite index normal subgroup $N$ of $G$ with $g \notin N$. Now we use a standard trick for graphs of residually finite groups with finite edge groups.

**Lemma 8.** *Let $\mathbb{A}$ be a finite graph of groups such that the vertex groups are residually finite SLI-groups and the edge groups are finite. Then the fundamental group of $\mathbb{A}$ is an SLI-group.*

*Proof.* Let $G$ be the fundamental group of $\mathbb{A}$. Then $G$ is residually finite [8]. Since there are only finitely many edge groups and each edge groups is finite, there is a finite index normal subgroup $N \leq G$ intersecting trivially each edge group, and hence each conjugate of an edge group. Thus the finitely generated subgroup $N \leq G$ acts on the Bass-Serre tree for $G$ [36] with trivial edge stabilizers, forcing $N$ to be a free product of conjugates of subgroups of the vertex groups of $G$ and a free group [36]. Since $N$ is finitely generated, these free factors must also be finitely generated. Since every finitely generated subgroup of an SLI-group is an SLI-group (Lemma 5) and $\mathbb{Z}$ is an SLI-group (Lemma 6), we may deduce that $N$ is a free product of SLI-groups and hence is an SLI-group by Lemma 7. Since $G$ contains $N$ as a finite index subgroup, Lemma 3 implies that $G$ is an SLI-group, as required. $\square$

Clearly, the trivial group $1$ is an SLI-group. Also all the defining properties of $\mathcal{C}$ preserve residual finiteness (the only non-trivial case being the graph of group constructions [8]). Hence, Lemma 4–6 and Lemma 8 immediately yield Theorem 1.

Our main application of Theorem 1 concerns graph groups:

**Theorem 2.** *The rational subset membership problem for a graph group $\mathbb{G}(\Sigma, I)$ is decidable if and only if $(\Sigma, I)$ is a transitive forest. Moreover, if $(\Sigma, I)$ is not a transitive forest, then there exists a fixed rational subset $L$ of $\mathbb{G}(\Sigma, I)$ such that the membership problem for $L$ within $\mathbb{G}(\Sigma, I)$ is undecidable.*

*Proof.* The decidability part follows immediately from Theorem 1: Lemma 2 implies that every graph group $\mathbb{G}(\Sigma, I)$ with $(\Sigma, I)$ a transitive forest belongs to the class $\mathcal{C}$.

Now assume that $(\Sigma, I)$ is not a transitive forest. By [37] it suffices to consider the case that $(\Sigma, I)$ is either a C4 or a P4. For the case of a C4 we can use Mihailova's result [30]. Now assume that $(\Sigma, I)$ is a P4. Let $\Sigma = \{a, b, c, d\}$ such that $(a, b) \in I$, $(b, c) \in I$, $(c, d) \in I$. In [1], Aalbersberg and Hoogeboom have shown that it is undecidable, whether $L \cap K = \emptyset$ for given rational trace languages $L, K \subseteq \mathbb{M}(\Sigma, I)$. In fact, the language $K$ is fixed, more precisely $K = ba(d(cb)^+a)^*dc^*$. The problem is that in construction of [1] the language $L$ is not fixed. This is due to the fact that Aalbersberg and Hoogeboom make a reduction from the undecidable problem whether a given 2-counter machine $C$ finally terminates, when initialized with empty counters. The pair of counter values $(m, n) \in \mathbb{N} \times \mathbb{N}$ is encoded by the single number $2^m 3^n$. It turns out that $K \cap L$ contains exactly those traces of the form $ab^{j_0} c^{j_1} dab^{j_1} c^{j_2} d \cdots ab^{j_{m-1}} c^{j_m} d$, such that $j_0 = 1$ and $C$ has a computation from the initial state to the final state in $m$ steps, where for every $0 \leq k \leq m$, $j_k$ is the encoding of the counter values after $k$ steps (note that $j_0 = 1$ indeed encodes the initial counter values $(0, 0)$).

Now let us choose for $C$ a fixed (universal) 2-counter machine such that it is undecidable whether $C$ finally terminates when started with the initial counter values $(m, n)$. Let $L \subseteq \mathbb{M}(\Sigma, I)$ be the *fixed* rational trace language constructed by Aalbersberg and Hoogeboom from $C$, and let us replace the fixed trace language $K = ba(d(cb)^+a)^*dc^*$ by the (non-fixed) language $K_{m,n} = b^{2^m 3^n} a(d(cb)^+a)^*dc^*$. Then it is undecidable, whether $K_{m,n} \cap L \neq \emptyset$ for given $m, n \in \mathbb{N}$. Hence, it is undecidable, whether $b^{-2^m 3^n} \in a(d(cb)^+a)^*dc^*L^{-1}$ in the graph group $\mathbb{G}(\Sigma, I)$. Clearly, $a(d(cb)^+a)^*dc^*L^{-1}$ is a fixed rational subset of the graph group $\mathbb{G}(\Sigma, I)$. □

We conclude this section with a further application of Theorem 1 to *graph products* (which should not be confused with graphs of groups). A graph product is given by a triple $(\Sigma, I, (G_v)_{v \in \Sigma})$, where $(\Sigma, I)$ is an independence alphabet and $G_v$ is a group, which is associated with the node $v \in \Sigma$. The group $\mathbb{G}(\Sigma, I, (G_v)_{v \in \Sigma})$ defined by this triple is the quotient

$$\mathbb{G}(\Sigma, I, (G_v)_{v \in \Sigma}) = *_{v \in \Sigma} G_v / \{xy = yx \mid x \in G_u, y \in G_v, (u, v) \in I\},$$

i.e., we take the free product $*_{v \in \Sigma} G_v$ of the groups $G_v$ ($v \in \Sigma$), but let elements from adjacent groups commute. Note that $\mathbb{G}(\Sigma, I, (G_v)_{v \in \Sigma})$ is the graph group $\mathbb{G}(\Sigma, I)$ in the case every $G_v$ is isomorphic to $\mathbb{Z}$. Graph products were first studied by Green [20].

**Theorem 3.** *If $(\Sigma, I)$ is a transitive forest and every group $G_v$ ($v \in V$) is finitely generated and virtually Abelian (i.e., has an Abelian subgroup of finite index), then the rational subset membership problem for $\mathbb{G}(\Sigma, I, (G_v)_{v \in \Sigma})$ is decidable.*

*Proof.* Assume that the assumptions from the theorem are satisfied. We show that $\mathbb{G}(\Sigma, I, (G_v)_{v \in \Sigma})$ belongs to the class $\mathcal{C}$. Since $(\Sigma, I)$ is a transitive forest, the group $\mathbb{G}(\Sigma, I, (G_v)_{v \in \Sigma})$ can be built up from trivial groups using the following two operations: (i) free products and (ii) direct products with finitely generated virtually Abelian groups. Since the class $\mathcal{C}$ is closed under free products, it suffices to prove that if $G$ belongs to the class $\mathcal{C}$ and $H$ is finitely generated virtually Abelian, then $G \times H$ also belongs to the class $\mathcal{C}$. As a virtually Abelian group, $H$ is a finite extension of a finite rank free Abelian group $\mathbb{Z}^n$. By the closure of the class $\mathcal{C}$ under direct products with $\mathbb{Z}$, $G \times \mathbb{Z}^n$ belongs to the class $\mathcal{C}$. Now, $G \times H$ is a finite extension $G \times \mathbb{Z}^n$, proving the theorem, since $\mathcal{C}$ is closed under finite extensions. □

## 4 The submonoid membership problem

Recall that the submonoid membership problem for a group $G$ asks for an algorithm to determine, given a group element $g \in G$ and a finitely generated submonoid $M$ of $G$, whether $g \in M$. Hence, there is a trivial reduction from the submonoid membership problem for $G$ to the rational subset membership problem for $G$. It turns out that there is also a reduction in the opposite direction, if we allow an additional free factor $\mathbb{Z}$ in the submonoid membership problem:

**Theorem 4.** *For every finitely generated group $G$, the rational subset membership problem for $G$ can be reduced to the submonoid membership problem for $G * \mathbb{Z}$. Moreover, if there exists a fixed rational subset $L \subseteq G$ such that the membership problem for $L$ within $G$ is undecidable, then there exists a fixed submonoid $M$ of $G * \mathbb{Z}$ such that the membership problem for $M$ within $G * \mathbb{Z}$ is undecidable.*

*Proof.* Let $\Sigma$ be a generating set for $G$ and let $h : (\Sigma^{\pm 1})^* \to G$ be the canonical morphism. Choose a generator $a \notin \Sigma$ for $\mathbb{Z}$. We also denote by $h$ the canonical morphism from $(\Sigma^{\pm 1} \cup \{a, a^{-1}\})^*$ to $G * \mathbb{Z}$.

Let $A = (Q, \Sigma^{\pm 1}, \delta, q_0, F)$ be a finite automaton and let $t \in (\Sigma^{\pm 1})^*$. By introducing $\varepsilon$-transitions, we may assume that the set of final states $F$ consists of a single state $q_f \neq q_0$. We will construct a finite subset $\Delta \subseteq (\Sigma^{\pm 1} \cup \{a, a^{-1}\})^*$ and a word $u \in (\Sigma^{\pm 1} \cup \{a, a^{-1}\})^*$ such that $h(t) \in h(L(A))$ if and only if $h(u) \in h(\Delta^*)$.

Without loss of generality assume that $Q = \{1, \ldots, n\}$. Choose an arbitrary generator $b \in \Sigma$ representing a non-trivial element of $G$ and define, for every $q \in \{1, \ldots, n\}$, the word $\widetilde{q}$ by

$$\widetilde{q} = a^q b a^{-q}$$

and let

$$\Delta = \{\widetilde{q}\, c\, \widetilde{p}^{-1} \mid (q, c, p) \in \delta\} \quad \text{and} \quad u = \widetilde{q_0}\, t\, \widetilde{q_f}^{-1}. \tag{7}$$

Note that in (7), we have $c \in \Sigma^{\pm 1} \cup \{\varepsilon\}$, since we introduced $\varepsilon$-transitions. We claim that $h(t) \in h(L(A))$ if and only if $h(u) \in h(\Delta^*)$.

Let us define a 1-*cycle* to be word of the form

$$\widetilde{q_1} v_1 \widetilde{q_2}^{-1}\, \widetilde{q_2} v_2 \widetilde{q_3}^{-1} \cdots \widetilde{q_{k-1}} v_{k-1} \widetilde{q_k}^{-1} \widetilde{q_k} v_k \widetilde{q_1}^{-1}$$

such that $k \geq 1$, $q_1, \ldots, q_k \in \{1, \ldots, n\}$, $v_1, \ldots, v_k \in (\Sigma^{\pm 1})^*$, and $h(v_1 \cdots v_k) = 1$. Note that a 1-cycle equals 1 in the free product $G * \mathbb{Z}$. We say that a word of the form $\widetilde{q_1} v_1 \widetilde{p_1}^{-1}\, \widetilde{q_2} v_2 \widetilde{p_2}^{-1} \cdots \widetilde{q_m} v_m \widetilde{p_m}^{-1}$, where $q_1, p_1, \ldots, q_m, p_m \in \{1, \ldots, n\}$ and $v_1, \ldots, v_m \in (\Sigma^{\pm 1})^*$, is 1-*cycle-free* if it does not contain a 1-cycle as a factor.

*Claim 1.* Let $m \geq 1$ and

$$v = \widetilde{q_1} v_1 \widetilde{p_1}^{-1}\, \widetilde{q_2} v_2 \widetilde{p_2}^{-1} \cdots \widetilde{q_m} v_m \widetilde{p_m}^{-1},$$

where $q_1, p_1, \ldots, q_m, p_m \in \{1, \ldots, n\}$, and $v_1, \ldots, v_m \in (\Sigma^{\pm 1})^*$. If $v = 1$ in $G * \mathbb{Z}$, then $v$ contains a 1-cycle.

*Proof of Claim 1.* We prove Claim 1 by induction over $m$. Assume that $v = 1$ in $G * \mathbb{Z}$. If $m = 1$, then $\widetilde{q_1} v_1 \widetilde{p_1}^{-1} = 1$ in $G * \mathbb{Z}$, i.e., $a^{q_1} b a^{-q_1} v_1 a^{p_1} b^{-1} a^{-p_1} = 1$ in $G * \mathbb{Z}$. If $v_1 \neq 1$ in $G$ then $a^{q_1} b a^{-q_1} v_1 a^{p_1} b^{-1} a^{-p_1}$ is irreducible and we obtain a contradiction. Now assume that $v_1 = 1$ in $G$. If $q_1 = p_1$, then $v$ is a single 1-cycle and we are ready. If $q_1 \neq p_1$ then we have $a^{q_1} b a^{p_1 - q_1} b^{-1} a^{-p_1} = 1$ in $G * \mathbb{Z}$. Since $p_1 - q_1 \neq 0$, the left-hand side of this identity is irreducible and we obtain again a contradiction.

Now assume that $m \geq 2$.

*Case 1.* There is $1 \leq i < m$ such that $p_i = q_{i+1}$. Then $v = 1$ in $G * \mathbb{Z}$ implies

$$\widetilde{q}_1 v_1 \widetilde{p}_1^{-1} \cdots \widetilde{q}_{i-1} v_{i-1} \widetilde{p}_{i-1}^{-1}\, \widetilde{q}_i (v_i v_{i+1}) \widetilde{p}_{i+1}^{-1}\, \widetilde{q}_{i+2} v_{i+2} \widetilde{p}_{i+2}^{-1} \cdots \widetilde{q}_m v_m \widetilde{p}_m^{-1} = 1$$

in $G * \mathbb{Z}$. By induction, we can conclude that the left-hand side of this identity contains a 1-cycle. But then also the word $v$ must contain a 1-cycle.

*Case 2.* $p_i \neq q_{i+1}$ for all $1 \leq i < m$. If there is $1 \leq i \leq m$ such that $v_i = 1$ in $G$ and $q_i = p_i$ then $v$ contains the 1-cycle $\widetilde{q}_i v_i \widetilde{p}_i^{-1}$. Now assume that $q_i \neq p_i$ whenever $v_i = 1$ in $G$. Remove from the word $v$ all factors $v_i$ with $v_i = 1$ in $G$ and let us call the resulting word $v'$. We claim that $v'$ is irreducible, when viewed as an alternating word in the free product $G * \mathbb{Z}$. For this consider a maximal subword of $v'$ of the form

$$\begin{aligned}
&\widetilde{p}_i^{-1} \widetilde{q}_{i+1} \widetilde{p}_{i+1}^{-1} \widetilde{q}_{i+2} \cdots \widetilde{p}_{j-1}^{-1} \widetilde{q}_j = \\
&a^{p_i} b^{-1} a^{-p_i} a^{q_{i+1}} b a^{-q_{i+1}} a^{p_{i+1}} b^{-1} a^{-p_{i+1}} a^{q_{i+2}} b a^{-q_{i+2}} \cdots a^{p_{j-1}} b^{-1} a^{-p_{j-1}} a^{q_j} b a^{-q_j} = \\
&a^{p_i} b^{-1} a^{q_{i+1}-p_i} b a^{p_{i+1}-q_{i+1}} b^{-1} a^{q_{i+2}-p_{i+1}} b a^{p_{i+2}-q_{i+2}} \cdots b^{-1} a^{q_j - p_{j-1}} b a^{-q_j}, \qquad (8)
\end{aligned}$$

where $j \geq i+1$, $v_{i+1} = \cdots = v_{j-1} = 1$ in $G$ and $v_i \neq 1 \neq v_j$ in $G$. Since $p_k \neq q_{k+1}$ for all $i \leq k \leq j-1$, each of the factors $a^{q_{k+1}-p_k}$ from (8) is non-trivial. The same is also true for the factors $a^{p_k - q_k}$ for $i+1 \leq k \leq j-1$, since $p_k \neq q_k$. It follows that the factor (8) of $v'$ is irreducible in the free product $G * \mathbb{Z}$. Similar arguments apply to the maximal prefix of $v'$ of the form

$$\widetilde{q}_1 \widetilde{p}_1^{-1} \widetilde{q}_2 \cdots \widetilde{p}_{j-1}^{-1} \widetilde{q}_j = a^{q_1} b a^{p_1 - q_1} b^{-1} a^{q_2 - p_1} b \cdots a^{p_{j-1} - q_{j-1}} b^{-1} a^{q_j - p_{j-1}} b a^{-q_j}, \quad (9)$$

and to the maximal suffix of the form

$$\begin{aligned}
&\widetilde{p}_i^{-1} \widetilde{q}_{i+1} \cdots \widetilde{p}_{m-1}^{-1} \widetilde{q}_m \widetilde{p}_m^{-1} = \\
&\quad a^{p_i} b^{-1} a^{q_{i+1}-p_i} b a^{p_{i+1}-q_{i+1}} b^{-1} \cdots a^{q_m - p_{m-1}} b a^{p_m - q_m} b^{-1} a^{-p_m}, \qquad (10)
\end{aligned}$$

and even to the whole word $v'$ in case $v_i = 1$ in $G$ for all $1 \leq i \leq m$.

Factors of the form (8)–(10) are separated in $v'$ with words $v_\ell \in (\Sigma^{\pm 1})^*$, where $v_\ell \neq 1$ in $G$. This shows that $v'$ is indeed a non-empty irreducible alternating word. But $v' = 1$ in $G * \mathbb{Z}$, which is a contradiction. This concludes the proof of Claim 1.

Now we can prove that $h(t) \in h(L(A))$ if and only if $h(u) = h(\widetilde{q}_0 t \widetilde{q}_f^{-1}) \in h(\Delta^*)$. First assume that $h(t) \in h(L(A))$. Let $a_1 \cdots a_m \in L(A)$ such that $(q_{i-1}, a_i, q_i) \in \delta$ for $1 \leq i \leq m$, $q_m = q_f$, and $h(a_1 \cdots a_m) = h(t)$. Then

$$h(\widetilde{q}_0 t \widetilde{q}_f^{-1}) = h(\widetilde{q}_0 a_1 \widetilde{q}_1^{-1} \widetilde{q}_1 a_2 \widetilde{q}_2^{-1} \cdots \widetilde{q}_{m-1} a_m \widetilde{q}_m^{-1}) \in h(\Delta^*).$$

Now assume that $h(\widetilde{q}_0 t \widetilde{q}_f^{-1}) \in h(\Delta^*)$. Thus,

$$\widetilde{q}_0 t \widetilde{q}_f^{-1} = \widetilde{q}_1 a_1 \widetilde{p}_1^{-1} \widetilde{q}_2 a_2 \widetilde{p}_2^{-1} \cdots \widetilde{q}_m a_m \widetilde{p}_m^{-1}$$

in $G * \mathbb{Z}$, where $q_1, p_1, \ldots, q_m, p_m \in \{1, \ldots, n\}$, $a_1, \ldots, a_m \in \Sigma^{\pm 1} \cup \{\varepsilon\}$, and $(q_i, a_i, p_i) \in \delta$ for $1 \leq i \leq m$. Without loss of generality we may assume that the word $\widetilde{q}_1 a_1 \widetilde{p}_1^{-1} \widetilde{q}_2 a_2 \widetilde{p}_2^{-1} \cdots \widetilde{q}_m a_m \widetilde{p}_m^{-1}$ is 1-cycle-free (otherwise we can remove all 1-cycles from this word; note that a 1-cycle equals 1 in the group $G * \mathbb{Z}$). Since

$$\underbrace{\widetilde{q}_f t^{-1} \widetilde{q}_0^{-1} \, \widetilde{q}_1 a_1 \widetilde{p}_1^{-1} \, \widetilde{q}_2 a_2 \widetilde{p}_2^{-1} \cdots \widetilde{q}_m a_m \widetilde{p}_m^{-1}}_{v} = 1,$$

in $G * \mathbb{Z}$, we know by Claim 1 that the word $v$ contains a 1-cycle. We claim that this 1-cycle must be the whole word $v$: first of all, the suffix $\widetilde{q}_1 a_1 \widetilde{p}_1^{-1} \cdots \widetilde{q}_m a_m \widetilde{p}_m^{-1}$ of $v$ is 1-cycle-free. If a prefix $\widetilde{q}_f t^{-1} \widetilde{q}_0^{-1} \, \widetilde{q}_1 a_1 \widetilde{p}_1^{-1} \cdots \widetilde{q}_i a_i \widetilde{p}_i^{-1}$ for $i < m$ is a 1-cycle, then $\widetilde{q}_{i+1} a_{i+1} \widetilde{p}_{i+1}^{-1} \cdots \widetilde{q}_m a_m \widetilde{p}_m^{-1} = 1$ in $G * \mathbb{Z}$. Hence, Claim 1 implies that the word $\widetilde{q}_{i+1} a_{i+1} \widetilde{p}_{i+1}^{-1} \cdots \widetilde{q}_m a_m \widetilde{p}_m^{-1}$ contains a 1-cycle, contradicting the fact that $\widetilde{q}_1 a_1 \widetilde{p}_1^{-1} \cdots \widetilde{q}_m a_m \widetilde{p}_m^{-1}$ is 1-cycle-free. Thus, indeed, $v$ is a 1-cycle. Hence, $q_0 = q_1$, $q_f = p_m$, $p_i = q_{i+1}$ for $1 \le i < m$, and $t^{-1} a_1 \cdots a_m = 1$ in $G$, i.e., $h(t) = h(a_1 \cdots a_m) \in h(L(A))$.

This concludes the reduction of the rational subset membership problem of $G$ to the submonoid membership problem for $G * \mathbb{Z}$. The second statement of Theorem 4 follows from the fact that $h(\Delta^*)$ is a fixed submonoid of $G * \mathbb{Z}$ if $A$ is a fixed finite automaton. □

Theorems 2 and 4 imply that the submonoid membership problem is undecidable for every graph group $\mathbb{G}(\Sigma \cup \{a\}, I)$, where $a \notin \Sigma$ and $(\Sigma, I)$ is not a transitive forest. In the rest of the paper, we will sharpen this result. We show that for a graph group the submonoid membership problem is decidable if and only if the rational subset membership problem is decidable, i.e., if and only if the independence alphabet is a transitive forest. In fact, by our previous results, it suffices to consider a P4:

**Theorem 5.** *Let $\Sigma = \{a, b, c, d\}$ and $I = \{(a, b), (b, c), (c, d)\}$, i.e, $(\Sigma, I)$ is a* P4*. Then there exists a fixed submonoid $M$ of $\mathbb{G}(\Sigma, I)$ such that the membership problem of $M$ within $\mathbb{G}(\Sigma, I)$ is undecidable.*

*Proof.* We follow the strategy of the proof of Theorem 4, but instead of arguing with alternating sequences in a free product, we have to argue with traces from $\mathbb{M}(\Sigma^{\pm 1}, I)$. Let $R$ denote the trace rewriting system over $\mathbb{M}(\Sigma^{\pm 1}, I)$ defined in (1). As usual let $h : (\Sigma^{\pm 1})^* \to \mathbb{G}(\Sigma, I)$ denote the canonical morphism, which will be identified with the canonical morphism $h : \mathbb{M}(\Sigma^{\pm 1}, I) \to \mathbb{G}(\Sigma, I)$. Let us fix a finite automaton $A$ over the alphabet $\Sigma^{\pm 1}$ such that the membership problem for $h(L(A))$ within $\mathbb{G}(\Sigma, I)$ is undecidable; such an automaton exists by Theorem 2. Assume that

$$A = (\{2, \ldots, n\}, \Sigma^{\pm 1}, \delta, q_0, \{q_f\}),$$

where $q_0 \neq q_f$ (it will be useful later that every state is a number greater than 1). For a state $q \in \{2, \ldots, n\}$ we define the trace $\widetilde{q} \in \mathbb{M}(\Sigma^{\pm 1}, I)$ by

$$\widetilde{q} = (ad)^q bc (ad)^{-q} = (ad)^q bc (d^{-1} a^{-1})^q.$$

Note that the dependence graph of $\widetilde{q}$ is almost a linear chain; only $b$ and $c$ in the middle may commute with each other. Moreover, every symbol from $\Sigma^{\pm 1}$ is dependent on $ad$, that is it does not commute with $ad$.

Let $\varphi : (\Sigma^{\pm 1})^* \to (\Sigma^{\pm 1})^*$ be the injective morphism defined by $\varphi(x) = xx$ for $x \in \Sigma^{\pm 1}$. Thus, $w \in L(A)$ if and only if $\varphi(w) \in \varphi(L(A))$. Since $(x, y) \in I$ implies that $\varphi(x)$ and $\varphi(y)$ commute, $\varphi$ can be lifted to a morphism $\varphi : \mathbb{M}(\Sigma^{\pm 1}, I) \to \mathbb{M}(\Sigma^{\pm 1}, I)$. The reader can easily verify that, for every trace $t \in \mathbb{M}(\Sigma^{\pm 1}, I)$, the equality $\mathrm{NF}_R(\varphi(t)) = \varphi(\mathrm{NF}_R(t))$ holds. In particular, $\varphi(t)$ is irreducible if and only if $t$ is irreducible and $h(t) = h(u)$ if and only if $h(\varphi(t)) = h(\varphi(u))$.

Let us fix a trace $t \in \mathbb{M}(\Sigma^{\pm 1}, I)$ and define

$$\Delta = \{\widetilde{q}\varphi(x)\widetilde{p}^{-1} \mid (q, x, p) \in \delta\} \subseteq \mathbb{M}(\Sigma^{\pm 1}, I) \text{ and } u = \widetilde{q_0}\varphi(t)\widetilde{q_f}^{-1} \in \mathbb{M}(\Sigma^{\pm 1}, I).$$

We claim that $h(t) \in h(L(A))$ if and only if $h(u) \in h(\Delta^*)$. For this, we can follow the proof scheme of Theorem 4. The following claim replaces Claim 1 from the previous proof.

*Claim 2.* Let $m \geq 1$ and

$$v = \widetilde{q_1}\varphi(v_1)\widetilde{p_1}^{-1}\ \widetilde{q_2}\varphi(v_2)\widetilde{p_2}^{-1} \cdots \widetilde{q_m}\varphi(v_m)\widetilde{p_m}^{-1},$$

where $q_1, p_1, \ldots, q_m, p_m \in \{2, \ldots, n\}$ and $v_1, \ldots, v_m \in (\Sigma^{\pm 1})^*$. If $v = 1$ in $\mathbb{G}(\Sigma, I)$, then $v$ contains a 1-cycle (1-cycles are defined as in the proof of Theorem 4).

*Proof of Claim 2.* Most parts of the proof can be copied from the proof of Claim 1. In fact, we only have to adapt those arguments from the proof of Claim 1, which were specific for the free product $G * \mathbb{Z}$. For the base case $m = 1$ we obtain the identity

$$(ad)^{q_1}bc(ad)^{-q_1}\varphi(v_1)(ad)^{p_1}b^{-1}c^{-1}(ad)^{-p_1} = 1 \tag{11}$$

in $\mathbb{G}(\Sigma, I)$. Assume without loss of generality that $v_1$, viewed as a trace, is irreducible with respect to $R$. Then also $\varphi(v_1)$ is irreducible. If $\varphi(v_1) = \varepsilon$, then (11) becomes

$$(ad)^{q_1}bc(ad)^{p_1-q_1}b^{-1}c^{-1}(ad)^{-p_1} = 1.$$

If $p_1 = q_1$, then $v$ is a 1-cycle. If $p_1 \neq q_1$, then we obtain a contradiction, since the trace $(ad)^{q_1}bc(ad)^{p_1-q_1}b^{-1}c^{-1}(ad)^{-p_1}$ is irreducible w.r.t. $R$. Now assume that $\varphi(v_1) \neq \varepsilon$. In the trace

$$(ad)^{q_1}bc(d^{-1}a^{-1})^{q_1}\varphi(v_1)(ad)^{p_1}b^{-1}c^{-1}(d^{-1}a^{-1})^{p_1}$$

only the last $a^{-1}$ of the factor $(d^{-1}a^{-1})^{q_1}$ may cancel against the first $a$ of $\varphi(v_1)$ (in case $a \in \min(v_1)$) and the first $a$ of the factor $(ad)^{p_1}$ may cancel against the last $a^{-1}$ of $\varphi(v_1)$ (in case $a^{-1} \in \max(v_1)$). To see this, note that if $a \in \min(v_1)$ then $\varphi(v_1) = aa\varphi(t)$ for some trace $t$. Then

$$(d^{-1}a^{-1})^{q_1}\varphi(v_1) = (d^{-1}a^{-1})^{q_1}aa\varphi(t) \to_R (d^{-1}a^{-1})^{q_1-1}d^{-1}a\varphi(t).$$

Since we assumed $a \in \min(v_1)$, the only other minimal element of the trace $a\varphi(t)$ may be $b$ or $b^{-1}$, both of which do not commute with $d^{-1}$. It follows that the trace $\mathrm{NF}_R((d^{-1}a^{-1})^{q_1}\varphi(v_1))$ is of the form $(d^{-1}a^{-1})^k d^{-1}a\varphi(t)$ for $k = q_1 - 1 \geq 1$ (since $q_1 \geq 2$). Moreover, if $a^{-1}$ is a maximal symbol of $t$, then $\varphi(t) = \varphi(t')a^{-1}a^{-1}$ for some trace $t'$. Hence, by making a possible cancellation with the first $a$ in $(ad)^{p_1}$, it follows finally that

$$\mathrm{NF}_R(\widetilde{q}_1\varphi(v_1)\widetilde{p}_1^{-1}) = (ad)^{q_1}bc(d^{-1}a^{-1})^k d^{-1}xd(ad)^\ell b^{-1}c^{-1}(d^{-1}a^{-1})^{p_1} \neq \varepsilon$$

for some trace $x$, where $\ell = p_1 - 1 \geq 1$. This contradicts again (11). This proves the inductive base case $m = 1$ in Claim 2.

For the inductive step for Claim 2, we can distinguish the same two cases as in the proof of Claim 1, where the first case can be treated exactly as in the proof of Claim 1. For the second case, we assume that $p_k \neq q_{k+1}$ for all $1 \leq k \leq m-1$ and $q_k \neq p_k$ whenever $v_k = 1$ in $\mathbb{G}(\Sigma, I)$. Let $v'$ be word that results from $v$ by deleting all factors $\varphi(v_i)$, which are equal 1 in $\mathbb{G}(\Sigma, I)$. In the following, we consider $v'$ as a trace. Consider a maximal factor of $v'$ of the form

$$\begin{aligned}
&\widetilde{p}_i^{-1}\widetilde{q}_{i+1}\widetilde{p}_{i+1}^{-1}\widetilde{q}_{i+2}\cdots\widetilde{p}_{j-1}^{-1}\widetilde{q}_j = \\
&(ad)^{p_i}b^{-1}c^{-1}(ad)^{-p_i}(ad)^{q_{i+1}}bc(ad)^{-q_{i+1}}(ad)^{p_{i+1}}b^{-1}c^{-1}(ad)^{-p_{i+1}}(ad)^{q_{i+2}}bc(ad)^{-q_{i+2}} \\
&\qquad \cdots (ad)^{p_{j-1}}b^{-1}c^{-1}(ad)^{-p_{j-1}}(ad)^{q_j}bc(ad)^{-q_j} = \\
&(ad)^{p_i}b^{-1}c^{-1}(ad)^{q_{i+1}-p_i}bc(ad)^{p_{i+1}-q_{i+1}}b^{-1}c^{-1}(ad)^{q_{i+2}-p_{i+1}}bc(ad)^{p_{i+2}-q_{i+2}} \\
&\qquad \cdots b^{-1}c^{-1}(ad)^{q_j-p_{j-1}}bc(ad)^{-q_j}, \qquad (12)
\end{aligned}$$

where $j \geq i+1$ and $\varphi(v_{i+1}) = \cdots = \varphi(v_{j-1}) = 1$, $\varphi(v_i) \neq 1 \neq \varphi(v_j)$ in $\mathbb{G}(\Sigma, I)$. The same arguments as in the proof of Claim 1 show that this trace is irreducible with respect to $R$, and similarly for the analogues of (9) and (10). In $v'$, factors of the form (12) are separated by traces $\varphi(v_i)$, where $\varphi(v_i) \neq 1$ in $\mathbb{G}(\Sigma, I)$. Without loss of generality assume that each such trace $\varphi(v_i)$ is irreducible and hence non-empty. As in the previous paragraph, for the base case $m = 1$, one can show that in such a concatenation, only a single minimal $a$ and a single maximal $a^{-1}$ of a trace $\varphi(v_i) \neq \varepsilon$ may be cancelled. It follows that $\mathrm{NF}_R(v) \neq \varepsilon$, which contradicts $v = 1$ in $\mathbb{G}(\Sigma, I)$. This concludes the proof of Claim 2. The rest of the argument is completely analogous to the proof of Theorem 4. □

Recall that a graph is not a transitive forest if and only if it either contains an induced C4 or P4 [37]. Together with Mihailova's result for the generalized word problem of $F(\{a,b\}) \times F(\{c,d\})$, Theorem 2 and 5 imply:

**Corollary 1.** *The submonoid membership problem for a graph group $\mathbb{G}(\Sigma, I)$ is decidable if and only if $(\Sigma, I)$ is a transitive forest. Moreover, if $(\Sigma, I)$ is not a transitive forest, then there exists a fixed submonoid $M$ of $\mathbb{G}(\Sigma, I)$ such that the membership problem for $M$ within $\mathbb{G}(\Sigma, I)$ is undecidable.*

Since P4 is a chordal graph, the generalized word problem for $\mathbb{G}(\mathsf{P4})$ is decidable [26]. Hence, $\mathbb{G}(\mathsf{P4})$ is an example of a group for which the generalized word problem is decidable but the submonoid membership problem is undecidable.

## 5 Open problems

The definition of the class $\mathcal{C}$ leads to the question whether decidability of the rational subset membership problem is preserved under direct products with $\mathbb{Z}$. An affirmative answer would lead in combination with the results from [25, 31] to a more direct proof of Theorem 1.

Concerning graph groups, the precise borderline for the decidability of the generalized word problem remains open. By [26], the generalized word problem is decidable if the independence alphabet is chordal. Since every transitive forest is chordal, Theorem 2 does not add any new decidable cases. On the other hand, if the independence alphabet contains an induced C4, then the generalized word problem is undecidable [30]. But it is open for instance, whether for a cycle of length 5 the corresponding graph group has a decidable generalized word problem.

Another open problem concerns the complexity of the rational subset membership problem for graph groups, where the independence alphabet is a transitive forest. If the independence alphabet is part of the input, then our decision procedure does not yield an elementary algorithm, i.e., an algorithm where the running time is bounded by an exponent tower of fixed height. This is due to the fact that each calculation of the Parikh image of a context-free language leads to an exponential blow-up in the size of the semilinear sets in the proof of Lemma 7. An NP lower bound follows from the NP-completeness of integer programming.

Theorem 4 leads to the question whether the decidability of the submonoid membership problem is preserved under free products (as it is the case for the generalized word problem and the rational subset membership problem). If this were true, then Theorem 4 would imply that the rational subset membership problem of a group $G$ can be reduced to the submonoid membership problem of $G$ (note that the submonoid membership problem of $\mathbb{Z}$ is decidable). Hence, for every group, the rational subset membership problem and the submonoid membership problem would be recursively equivalent.

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