Hamiltonicity of automatic graphs

Dietrich Kuske and Markus Lohrey

Abstract It is shown that the existence of a Hamiltonian path in a planar automatic graph of bounded degree is complete for Σ_1^1, the first level of the analytical hierarchy. This sharpens a corresponding result of Hirst and Harel for highly recursive graphs. Furthermore, we also show: (i) The Hamiltonian path problem for finite planar graphs that are succinctly encoded by an automatic presentation is NEXPTIME-complete. (ii) The existence of an infinite path in an automatic successor tree is Σ_1^1-complete. (iii) An infinite version of the set cover problem is decidable for automatic graphs (it is Σ_1^1-complete for recursive graphs).

1 Introduction

The theory of *recursive structures* has its origins in computability theory. A structure is recursive, if its domain is a recursive set of naturals, and every relation is again recursive. Starting with the work of Manaster and Rosenstein [23] and Bean [1, 2], infinite variants of classical graph problems for finite graphs were studied for recursive graphs. It is not surprising that these problems are mostly undecidable for recursive graphs. This motivates the search for the precise level of undecidability. It turned out that some of the problems reside on low levels of the arithmetic hierarchy (e.g. the question whether a given recursive graph has an Eulerian path [3]), whereas others are complete for Σ_1^1 — the first level of the analytic hierarchy [21]. A classical example for the latter situation is the question whether a given recursive tree has an infinite path. With a technically quite subtle reduction from the latter problem, Harel proved in [13] that also the existence of a *Hamiltonian path* (i.e., a one-way infinite path that visits every node exactly once) in a recursive graph is Σ_1^1-complete. Σ_1^1-

Dietrich Kuske
Institut für Informatik, Universität Leipzig, e-mail: kuske@informatik.uni-leipzig.de

Markus Lohrey
Institut für Informatik, Universität Leipzig, e-mail: kuske@informatik.uni-leipzig.de

hardness holds already for highly recursive graphs, where a list of the neighbors of a node *v* can be computed effectively from *v*.

Hamiltonian paths in infinite graphs were also studied under a purely graph theoretic view. An important result of Dean, Thomas, and Yu [6] states that an infinite undirected graph *G* has an Hamiltonian path if it is (i) planar, (ii) 4-connected, and (iii) has only one end (see [7] for definitions). This extends a result of Tutte [27] for finite graphs.

In computer science, in particular in the area of automatic verification, focus has shifted in recent years from arbitrary recursive graphs to subclasses that have more amenable algorithmic properties. An important example for this is the class of *automatic graphs* [5, 16]. A graph is called automatic if it has an *automatic presentation*, which consists of a finite automaton that generates the set of nodes and a two-tape automaton with synchronously moving heads, which accepts the set of edges. One of the main motivations for investigating automatic graphs is the fact that every automatic graph has a decidable first-order theory [16], this result extends to first-order logic with infinity and modulo quantifiers [5, 19]. In contrast to these positive results, Khoussainov, Nies, and Rubin have shown that the isomorphism problem for automatic graphs is Σ_1^1-complete [17]. Results on the model theoretic complexity of automatic structures can be found in [15].

The main result of this paper states that the existence of a Hamiltonian path becomes Σ_1^1-complete already for a quite restricted subclass of recursive graphs, namely for automatic graphs, which are planar and of bounded degree. The latter means that there exists a constant *c* such that every node has at most *c* many neighbors. The proof of the Σ_1^1 lower bound (the non-trivial part) in Section 3 is based on a reduction from the *recurring tiling problem* [10, 12]. This is a variant of the classical tiling problem [29, 4] that asks whether a given finite set of tiles allows a tiling of the infinite quarter plane such that a distinguished color occurs infinitely often at the lower border. Harel proved that the recurring tiling problem is Σ_1^1-complete [10, 12]. In our reduction we use as building blocks some of the graph gadgets from the NP-hardness proof of the Hamiltonian path problem in finite planar graphs [9]. These gadgets have to be combined in a non-trivial way for the whole reduction.

The main purpose of automatic presentations is the finite representation of infinite structures. But automatic presentations can be also used as a tool for the succinct representation of large finite structures. An automatic presentation of size *n* may generate a finite graph of size $2^{O(n)}$. A straightforward adaptation of our proof for infinite automatic graphs shows that it is NEXPTIME-complete to check whether a finite planar graph given by an automatic presentation has a Hamiltonian path, see Section 4. Without the restriction to planar graphs, this result was already shown by Veith [28] in the slightly different context of graphs represented by ordered binary decision diagrams (OBDDs). The special OBDDs considered by Veith in [28] can be seen as automatic presentations of finite graphs.

Finally, in Section 5 we investigate some other graph problems in the automatic setting. Using a proof technique from [20, 15], we prove that the fundamental Σ_1^1-complete problem in recursion theory, namely the existence of an infinite path in a recursive tree remains Σ_1^1-complete if the input tree is automatic. For this result it is crucial that the tree is a *successor tree*, which means that it is an acyclic graph, where every node is reachable from a root node and every node except the root has exactly one incoming edge. If trees are given as particular partially ordered sets (order trees), then the existence of an infinite path is decidable for automatic trees [20].

From the above results, one might get the feeling that graph problems always have the same degree of undecidability in the recursive and in the automatic world. To the contrary, there are problems that are Σ_1^1-complete for recursive graphs [14] but decidable for automatic graphs. This applies to the existence of an infinite branch in an automatic *order tree* (i.e., the reflexive and transitive closure of a successor tree, Khoussainov, Rubin, and Stephan [20]) as well as to the existence of an infinite clique in an automatic graph (Rubin [25]). We show that also an infinite version of the set cover problem is decidable for automatic graphs. This result is achieved by providing a decision procedure for a fragment of second-order logic that allows to express the set cover problem as well as the two other decidable problems mentioned before.

Proofs, which are not included in this extended abstract will appear in the long version of this paper.

2 Preliminaries

Infinite graphs and Hamiltonian paths

For details on graph theory see [7]. A *graph* is a pair $G = (V, E)$, where V is the (possibly infinite) set of nodes and $E \subseteq V \times V$ is the set of edges. It is *undirected* if $(u, v) \in E$ implies $(v, u) \in E$. The graph G has *degree at most c*, where $c \in \mathbb{N}$, if every node is contained in at most c many edges. If G has degree at most c for some constant c, then G has *bounded degree*. If it is only required that every node is contained in only finitely many edges then G is called *locally finite*. The graph G is *planar* if it can be embedded in the Euclidean plane without crossing edges and without accumulation points; any such embedding is a *plane graph*. A *finite path* in G is a sequence $[v_1, v_2, \dots, v_n]$ of nodes such that $(v_i, v_{i+1}) \in E$ for all $1 \le i \le n$. The nodes v_1 and v_n are the end points of this path. The graph $G = (V, E)$ is *connected* if for all $u, v \in V$ there exists a finite path in the undirected graph $(V, E \cup \{(x, y) \mid (y, x) \in E\}$ with end points u and v. An *infinite path* in G is an infinite sequence $[v_1, v_2, \dots]$ such that every initial segment is a finite path. A *Hamiltonian path* (or *spanning ray*) of an *infinite* graph G is an infinite path $[v_1, v_2, \dots]$ in G that visits every node of G exactly once, i.e. the mapping $i \mapsto v_i$ ($i \in \mathbb{N}$) is a bijection between $\mathbb{N}$ and the set of nodes.

Recursive graphs and automatic graphs

A *recursive graph* is a graph $G = (V,E)$ such that V and E are recursive subsets of $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$, respectively. In case G is infinite, one can w.l.o.g. assume that $V = \mathbb{N}$. A recursive graph G is *highly recursive* if it is locally finite and for every node v a list of its finitely many neighbors can be computed from v. Harel [13] has shown the following result:

Theorem 1 ([13]). *It is Σ_1^1-complete to determine, whether a given highly recursive undirected graph of bounded degree has a Hamiltonian path.*

Recall that Σ_1^1 is the first level of the *analytic hierarchy* [21]. More precisely, it is the class of all subsets of $\mathbb{N}$ of the form $\{n \in \mathbb{N} \mid \exists A\, \varphi(A)\}$, where $\varphi(A)$ is a formula of first-order arithmetic. In Thm. 1, a recursive graph is encoded by a pair of Gödel numbers for machines for the node and edge set, respectively.

In [14], Hirst and Harel proved that for planar recursive graphs the existence of a Hamiltonian path is still Σ_1^1-complete. The aim of this paper is to extend the results from [13, 14] to the class of planar automatic graphs of bounded degree. We introduce this class of graphs briefly, more details can be found in [16, 5]

Let us fix $n \in \mathbb{N}$ and a finite alphabet Γ. Let $\# \notin \Gamma$ be an additional padding symbol. For words $w_1, \dots, w_n \in \Gamma^*$ we define the *convolution* $w_1 \otimes w_2 \otimes \cdots \otimes w_n$, which is a word over the alphabet $\prod_{i=1}^{n}(\Gamma \cup \{\#\})$, as follows: Let $w_i = a_{i,1}a_{i,2}\cdots a_{i,k_i}$ with $a_{i,j} \in \Gamma$ and $k = \max\{k_1, \dots, k_n\}$. For $k_i < j \leq k$ define $a_{i,j} = \#$. Then $w_1 \otimes \cdots \otimes w_n = (a_{1,1}, \dots, a_{n,1}) \cdots (a_{1,k}, \dots, a_{n,k})$. Thus, for instance $aba \otimes bbabb = (a,b)(b,b)(a,a)(\#,b)(\#,b)$. An n-ary relation $R \subseteq (\Gamma^*)^n$ is called automatic if the language $\{w_1 \otimes \cdots \otimes w_n \mid (w_1, \dots, w_n) \in R\}$ is a regular language.

Now let $\mathscr{A} = (A, (R_i)_{i \in J})$ be a relational structure with finitely many relations, where $R_i \subseteq A^{n_i}$. A tuple (Γ, L, h) is called an *automatic presentation* for $\mathscr{A}$ if (i) Γ is a finite alphabet, (ii) $L \subseteq \Gamma^*$ is a regular language, (iii) $h : L \to A$ is a bijective function, (iv) the relation $\{(u,v) \in L \times L \mid h(u) = h(v)\}$ is automatic, and (v) the relation $\{(u_1, \dots, u_{n_i}) \in L^{n_i} \mid (h(u_1), \dots, h(u_{n_i})) \in R_i\}$ is automatic for every $i \in J$. We say that $\mathscr{A}$ is *automatic* if there exists an automatic presentation for $\mathscr{A}$. In the rest of the paper we will mainly restrict to automatic graphs. Such a graph can be represented by an automaton for the node set and an automaton for the edge set. Clearly, a (locally finite) automatic graph is (highly) recursive.

In contrast to recursive graphs, automatic graphs have some nice algorithmic properties. In [16] it was shown that the first-order theory of an automatic structure is decidable. This result extends to first-order logic with infinity and modulo quantifiers [5, 19]. For general automatic structures, these logics do not allow elementary algorithms [5]. On the other hand, for automatic structures with a Gaifman graph of bounded degree first-order logic extended by a rather general class of counting quantifiers can be decided in triply exponential space [22].

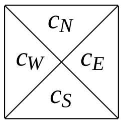
In contrast to these positive results, several strong undecidability results show that algorithmic methods for automatic structures are quite limited. Since the configuration graph of a Turing machine is automatic, it follows easily that reachability in automatic graphs is undecidable. Khoussainov, Nies, and Rubin have shown that the isomorphism problem for automatic graphs is Σ_1^1-complete [17], whereas isomorphism of locally finite automatic graphs is Π_3^0-complete [24]. Our main result is the following:

Theorem 2. *It is Σ_1^1-complete to determine, whether a given planar automatic undirected graph of bounded degree has a Hamiltonian path.*

Note that the Σ_1^1 upper bound in Thm. 2 follows immediately from the corresponding result for general recursive graphs (Thm. 1). For the lower bound we use a special variant of the tiling problem [29, 4] that was introduced by Harel.

Tilings

Our main tool for proving Σ_1^1-hardness of the existence of a Hamiltonian path in a planar automatic graph of bounded degree is the *recurring tiling problem* [10, 12]. An instance of the recurring tiling problem consists of (i) a finite set of *colors* $C = \{c_0, c_1, \ldots, c_n\}$, (ii) a distinguished color c_0, and (iii) a set $\mathscr{T} \subseteq C^4$ of *tile types*. For a tile type $t \in \mathscr{T}$ we write $t = (t_W, t_N, t_E, t_S)$ ("W" for west, "N" for north, "E" for east "S" for south); a visualization looks as follows:



A mapping $f : \mathbb{N}^2 \to \mathscr{T}$ is a *tiling* if, for every $(i, j) \in \mathbb{N}^2$, we have $f(i, j)_N = f(i+1, j)_S$ and $f(i, j)_E = f(i, j+1)_W$. A *recurring tiling* is a tiling f such that for infinitely many $j \in \mathbb{N}$, we have $f(0, j)_S = c_0$. Now the recurring tiling problem asks whether a given problem instance has a recurring tiling. Harel has shown the following result:

Theorem 3 ([10]). *The recurring tiling problem is Σ_1^1-complete.*

The recurring tiling problem turned out be very useful for proving Σ_1^1 lower bounds for certain satisfiability problems in logic [11].

3 Hamiltonicity for automatic graphs

In this section, we reduce the recurring tiling problem to the existence of a Hamiltonian path in a planar automatic graph of bounded degree. This proves Thm. 2 by Thm. 3.

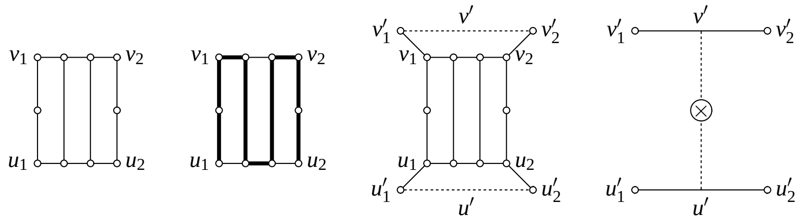


Fig. 1 The graph X, its use and abbreviation

3.1 Building blocks

Let us introduce several building blocks from which we assemble our final planar automatic graph of bounded degree. These building blocks are variants of graphs taken from the NP-hardness proof for the Hamiltonian path problem in finite planar graphs [9].

Exclusive or

Consider the finite plane graph X in Fig. 1 (first picture). It has a Hamiltonian path from u_1 to u_2 (and similarly from v_1 to v_2) indicated in the second picture. Now suppose G' is some graph containing the edges u' and v'. Then we build a graph G as follows: in the disjoint union of G' and X, delete the edges u' and v' and connect their endpoints to u_1 and u_2 (to v_1 and v_2, resp., see Fig. 1, third picture). Now suppose H is a Hamiltonian path in G with no endpoint in X. Suppose u_1 is the first vertex from X in H. Then the restriction of H to X has to coincide with the Hamiltonian path from u_1 to u_2. Hence H gives rise to a Hamiltonian path in G' that coincides with H on G' but passes through the edge u' instead of taking the detour through X. Note that H' does not contain the edge v'. Conversely, every Hamiltonian path H' of G' that contains the edge u' but not the edge v' induces a Hamiltonian path H of G in a similar way. Joining X to the graph G' in this manner restricts the Hamiltonian paths to those that either contain the edge u' or the edge v', but not both. This also explains the name X: this graph acts as an “exclusive-or”. Note that, if G' is planar and the two edges u' and v' belong to the same face, then also G can be constructed as a planar graph. Since we will make repeated use of this construction, we abbreviate it as in Fig. 1, fourth picture.

Boolean functions

Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function. In the NP-hardness proof of [9], a planar graph G together with distinguished edges $e_1, \dots, e_n$ is constructed such that $f(b_1, \dots, b_n) = 1$ iff G has a Hamiltonian cycle H with $b_i = 1 \Leftrightarrow e_i \in H$. We modify

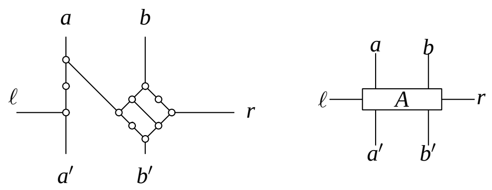


Fig. 2 The graph A and its abbreviation

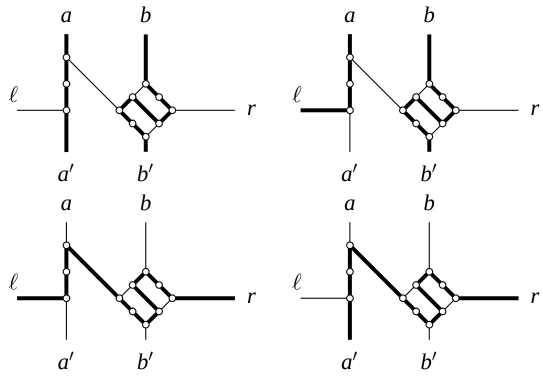


Fig. 3 Paths through the graph A

this construction slightly in order to place the edges e_i and two vertices u and v in a specified order at the boundary of the outer face.

Theorem 4. *There exists a constant c such that from given $k, \ell, n \in \mathbb{N}$ and $F \subseteq 2^{\{1,\ldots,k+\ell+n\}}$, one can construct effectively a finite plane graph G_F of degree at most c such that:*

- *At the boundary of the outer face of G_F, we find (in this counter-clockwise order) edges $e_1, \ldots e_k$, a vertex u, edges $e_{k+1}, \ldots, e_{k+\ell}$, a vertex v, and edges $e_{k+\ell+1}, \ldots, e_{k+\ell+n}$.*
- *For every $M \subseteq \{1, \ldots, k+\ell+n\}$, $M \in F$ iff there is a Hamiltonian path H from u to v such that $M = \{i \mid e_i$ belongs to $H\}$.*

Infinity checking

Next consider Fig. 2 – it depicts a graph A that is connected to some context via the edges ℓ, a, a', b, b', and r. If the complete graph has a Hamiltonian path, then locally, it has to be of one of the four forms depicted in Fig. 3.

Now consider Fig. 4 – it consists of infinitely many copies of the graph A arranged in a line, the edges a' and b' connect these copies of A with a line of nodes. Suppose the edges a and b of the copies of A are connected to some infinite graph G. Then, every Hamiltonian path H of the resulting graph has to enter and leave

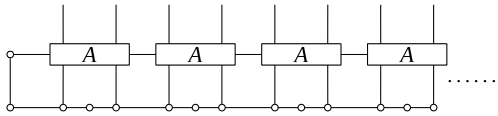


Fig. 4 The infinite graph L

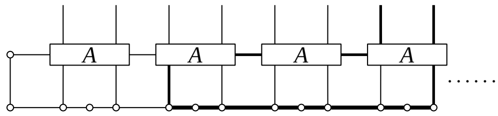


Fig. 5 A visit of a Hamiltonian path to the graph L

L infinitely often. Since the possibilities to pass A are restricted as shown in Fig.3, any such visit has to look as described in Fig. 5, i.e., the path enters from a into some copy of A, moves left to some copy of A (possibly without doing any step), moves down to the third line where it goes all the way back until it can enter the first A-copy via the edge b' and leave it via the edge b.

3.2 Assembling

From an instance of the recurring tiling problem, we construct in this section a planar automatic graph G of bounded degree that has an Hamiltonian path iff the instance of the recurring tiling problem admits a solution. So, we fix a finite set $C = \{c_0, c_1, \ldots, c_n\}$ of colors, a distinguished color c_0, and a set $\mathscr{T} \subseteq C^4$ of tile types. Next let

$$\mathscr{V} = \{W_0, W_1, \ldots, W_n, S_0, S_1, \ldots, S_n, \overline{N_0}, \overline{N_1}, \ldots, \overline{N_n}, \overline{E_0}, \overline{E_1}, \ldots, \overline{E_n}\}.$$

We will describe tile types by certain subsets of $\mathscr{V}$ where W_i expresses that the left color is c_i, and $\overline{N_i}$ denotes that the top color is *not* c_i (S_i and $\overline{E_i}$ refer to the bottom and right color and are to be understood similarly). More precisely, the tile $d = (c_i, c_j, c_k, c_\ell)$ is denoted by the set $\mathbb{S}_d = \{W_i\} \cup \{\overline{N_m} \mid m \neq j\} \cup \{\overline{E_m} \mid m \neq k\} \cup \{S_\ell\}$. Now let $F = \{\mathbb{S}_d \mid d \in \mathscr{T}\}$ be the descriptions of all the tile types d in $\mathscr{T}$. Then, by Thm. 4, there are finite plane graphs G_1, G_2, G_3, and G_4 with the following properties: (i) at the outer face, we find edges e for $e \in \mathscr{V}$ and nodes u and v in the order indicated in Fig. 6 and (ii) $M \in F$ iff there exists a Hamiltonian path H of G_x from u to v such that $M = \{v \in \mathscr{V} \mid v \text{ belongs to } H\}$ (for all $1 \leq x \leq 4$ and $M \subseteq \mathscr{V}$).

Next we choose mutually disjoint graphs $G(k, \ell)$ (for $k, \ell \in \mathbb{N}$) such that

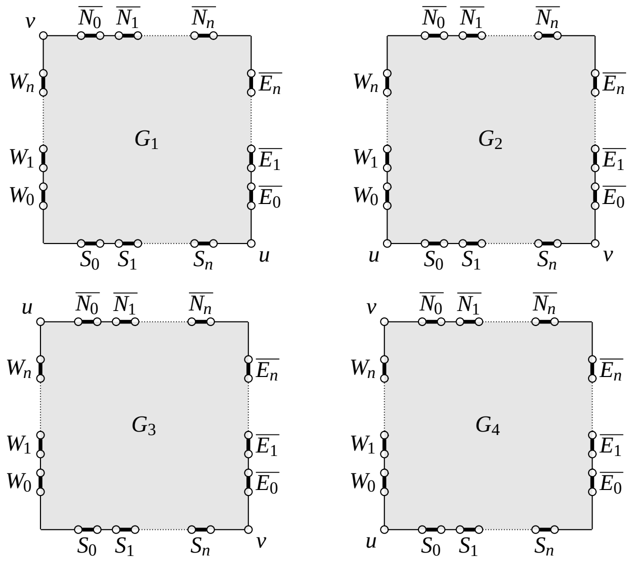


Fig. 6 The graphs G_x

$$G(k,\ell) \cong \begin{cases} G_1 & \text{if } k+\ell \text{ is even and } k>0 \text{ or } k=\ell=0 \\ G_2 & \text{if } k+\ell \text{ is odd and } \ell=0 \\ G_3 & \text{if } k+\ell \text{ is odd and } \ell>0 \\ G_4 & \text{if } k+\ell \text{ is even, } k=0, \text{ and } \ell>0. \end{cases}$$

Then $u(k,\ell)$ and $v(k,\ell)$ refer to the nodes u and v of the graph $G(k,\ell)$; similarly, $e(k,\ell)$ for $e \in \mathscr{V}$ refers to the edge e of the graph $G(k,\ell)$. In the disjoint union of these graphs $G(k,\ell)$, we connect the node $v(k,\ell)$ by a new edge with the following node:

$$\begin{aligned} u(k+1,\ell) &\text{ for } k+\ell \text{ even and } \ell=0 \\ u(k+1,\ell-1) &\text{ for } k+\ell \text{ even and } \ell>0 \\ u(k-1,\ell+1) &\text{ for } k+\ell \text{ odd and } k>0 \\ u(k,\ell+1) &\text{ for } k+\ell \text{ odd and } k=0. \end{aligned}$$

The result G^1 of this construction is visualized in Fig. 7 where the vertices $u(k,\ell)$ are denoted by empty nodes and $v(k,\ell)$ by filled nodes. From G^1 we construct G^2 by replacing the edges $\overline{E_i}(k,\ell)$ and $W_i(k,\ell+1)$ as well as $\overline{N_i}(k,\ell)$ and $S_i(k+1,\ell)$ ($k,\ell \in \mathbb{N}$, $0 \leq i \leq n$) by a copy of the exclusive-or graph X, see Fig. 8. In a third step, we construct G^3 by adding to G^2 the graph L from Fig. 4. To connect L to G^2, the start node of the edges a and b, resp., of the i^{th} copy of A in L is the left and right, resp., node of the edge $S_0(0,i)$. The final graph G is obtained from G^3 by adding a new node $\perp$ together with an edge between $\perp$ and $u(0,0)$.

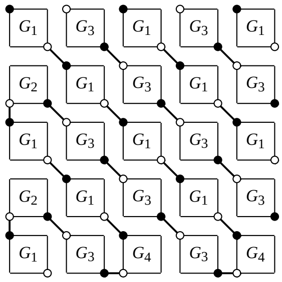


Fig. 7 First step in global construction - the graph G^1

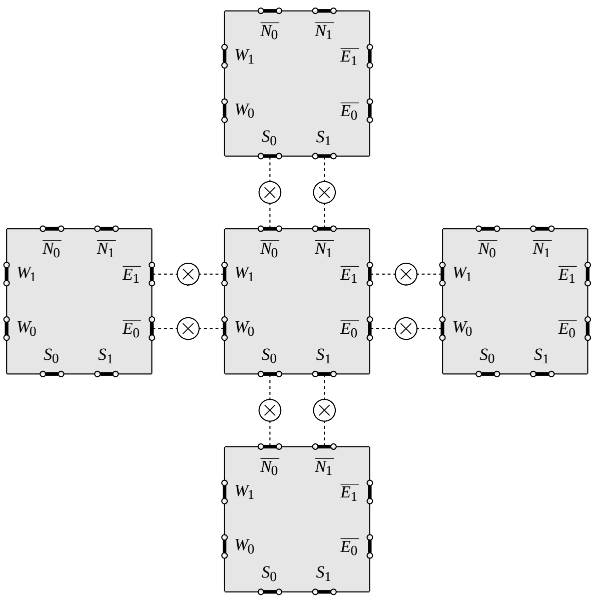


Fig. 8 Second step in global construction – the graph G^2 (for two colors c_0 and c_1)

Let us now prove that G has a Hamiltonian path iff $\mathscr{T}$ admits a recurring tiling. First suppose there is a recurring tiling $f : \mathbb{N} \times \mathbb{N} \to \mathscr{T}$. Let $k, \ell \in \mathbb{N}$ and $f(k,\ell) = (c_W, c_N, c_E, c_S)$. Then the graph $G(k,\ell) \in \{G_x \mid 1 \le i \le 4\}$ has a Hamiltonian path $H(k,\ell)$ from $u(k,\ell)$ to $v(k,\ell)$ such that for all $1 \le i \le n$

1. the edge S_i belongs to $H(k,\ell)$ iff $c_S = c_i$,
2. the edge W_i belongs to $H(k,\ell)$ iff $c_W = c_i$,
3. the edge $\overline{N_i}$ belongs to $H(k,\ell)$ iff $c_N \neq c_i$, and
4. the edge $\overline{E_i}$ belongs to $H(k,\ell)$ iff $c_E \neq c_i$.

Then we find a Hamiltonian path H_1 of the infinite graph G^1 in Fig. 7 by appending these Hamiltonian paths suitably:

$$H_1 = H(0,0), H(1,0), H(0,1), H(0,2), H(1,1), H(2,0)\dots$$

Since f is a tiling, we get

$$\begin{aligned}\overline{E_i}(k,\ell) \notin H_1 &\iff f(k,\ell)_E = c_i\\ &\iff f(k,\ell+1)_W = c_i\\ &\iff W_i(k,\ell+1) \in H_1\end{aligned}$$

and similarly $\overline{N_i}(k,\ell) \notin H_1$ iff $S_i(k+1,\ell) \in H_1$. Hence the Hamiltonian path H_1 can be extended to a Hamiltonian path H_2 of the graph G^2 obtained from G^1 by adding all the copies of the exclusive-or graph X. Observe also that f is recurring, i.e., there are infinitely many $\ell \in \mathbb{N}$ with $f(0,\ell)_S = c_0$. For every such ℓ, the path H_1 passes through the edge $S_0(0,\ell)$. Instead of passing through this edge, we now enter the graph L (Fig. 4) via the edge a of the ℓ^{th} copy of A and leave it via its edge b. We can ensure that after this visit, all nodes of L to the left of the ℓ^{th} copy of A have been visited (cf. Fig. 5). This results in a Hamiltonian path H_3 of the graph G^3 starting in $u(0,0)$. Prepending the node $\perp$ gives a Hamiltonian path H of the final graph G.

Conversely, let H be a Hamiltonian path of the final graph G. Since $\perp$ has degree 1, it has to start in $\perp$ – deleting $\perp$ from H gives a Hamiltonian path H_3 of G^3 that starts in $u(0,0)$. Since G^3 contains infinitely many nodes outside of L, this path has to enter and leave L infinitely often. Any such visit has to enter via the edge a some copy of A and leave via the edge b of the same copy of A (or vice versa, see Fig. 5). Hence, deleting all the vertices of L from the path H, we obtain a Hamiltonian path H_2 of the graph G^2 that contains infinitely many edges of the form $S_0(0,\ell)$. Recall that G^2 is obtained from G^1 by replacing some pairs of edges by the exclusive-or graph X. Hence, the restriction of H_2 to the nodes of G^1 gives rise to a Hamiltonian path H_1 of G^1 that

(a) contains infinitely many edges of the form $S_0(0,\ell)$,
(b) contains the edge $W_i(k,\ell+1)$ iff it does not contain the edge $\overline{E_i}(k,\ell)$, and
(c) contains the edge $S_i(k+1,\ell)$ iff it does not contain the edge $\overline{N_i}(k,\ell)$

for all $0 \le i \le n$ and $k,\ell \in \mathbb{N}$. Since H_1 has to pass through all the graphs $G(k,\ell)$, it has to be of the form

$$H(0,0), H(1,0), H(0,1), H(0,2), H(1,1), H(2,0)\dots$$

where $H(k,\ell)$ is a Hamiltonian path of the graph $G(k,\ell)$ from $u(k,\ell)$ to $v(k,\ell)$. Now we are ready to define the mapping $f : \mathbb{N}^2 \to C^4$: set

(1) $f(k,\ell)_W = c_i$ iff $H(k,\ell)$ contains the edge $W_i(k,\ell)$,
(2) $f(k,\ell)_N = c_i$ iff $H(k,\ell)$ does not contain the edge $\overline{N_i}(k,\ell)$,
(3) $f(k,\ell)_E = c_i$ iff $H(k,\ell)$ does not contain the edge $\overline{E_i}(k,\ell)$, and
(4) $f(k,\ell)_S = c_i$ iff $H(k,\ell)$ contains the edge $S_i(k,\ell)$.

Since $H(k,\ell)$ is a Hamiltonian path of $G(k,\ell)$ from $u(k,\ell)$ to $v(k,\ell)$, we get $f(k,\ell) \in \mathscr{T}$ from the construction of the graphs G_1, G_2, G_3, G_4. By (1), (b), and (3), we have

$$\begin{aligned} f(k,\ell)_W = c_i &\iff W_i(k,\ell) \text{ belongs to } H(k,\ell) \\ &\iff \overline{E_i}(k,\ell+1) \text{ does not belong to } H(k,\ell+1) \\ &\iff f(k,\ell+1) = c_i \end{aligned}$$

and similarly $f(k,\ell)_N = f(k+1,\ell)_S$ follows from (2), (c), and (4). Thus, f is a tiling. Since H_1 contains infinitely many edges of the form $S_0(0,\ell)$, there are infinitely many $\ell \in \mathbb{N}$ such that $S_0(0,\ell)$ belongs to $H(0,\ell)$, i.e., $f(0,\ell)_S = c_0$.

Thus, we showed that indeed the graph G contains a Hamiltonian path iff the set of tiles $\mathscr{T}$ admits a recurring tiling.

Clearly, the undirected graph G is planar and has bounded degree. Thus, in order to finish the proof of Thm. 2, it remains to prove that G is automatic. Note that the graph G has a highly regular structure. It results from the infinite grid $\mathbb{N} \times \mathbb{N}$ by replacing each grid point by a finite graph and connecting these finite graphs in a regular pattern. It is not surprising that such a graph is automatic, in particular since the grid is automatic. Let us provide some more formal arguments for the automaticity of G.

Recall that G can be obtained from $\mathbb{N} \times \mathbb{N}$ by replacing every grid point $(k,\ell) \in \mathbb{N} \times \mathbb{N}$ by a finite graph $G'(k,\ell)$. This graph is a copy of one of the graphs G'_1, G'_2, G'_3, G'_4, where G'_i is the graph G_i together with copies of the XOR-graph X that connect $G(k,\ell)$ with $G(k+1,\ell)$ and $G(k,\ell+1)$. Whether $G'(k,\ell)$ is G'_i only depends on the parity of $k+\ell$ and whether k and ℓ are zero or non-zero, respectively.

The alphabet of our presentation consists of the elements of $\{0,1,\#\}^2 \setminus \{(\#,\#)\}$ and the nodes of the graphs $G'_1, \dots, G'_4$. Then, the node set of G can be represented by the regular language

$$\{(\mathrm{bin}(k) \otimes \mathrm{bin}(\ell))\, v \mid k,\ell \geq 0,\ v \text{ is a node of } G'(k,\ell)\}, \tag{1}$$

where $\mathrm{bin}(n)$ is the binary encoding of a number n (note that the parity of $k+\ell$ can be determined by a finite automaton from $\mathrm{bin}(k) \otimes \mathrm{bin}(\ell)$). Constructing from this node representation an automaton that recognizes the edge set of G is straightforward but tedious. This concludes the proof of Thm. 2.

There also exists the variant of two-way Hamiltonian paths in infinite graphs. A two-way Hamiltonian path in $G = (V,E)$ is a two-way infinite sequence $(v_i)_{i \in \mathbb{Z}}$ such that $(v_i, v_{i+1}) \in E$ for all $i \in \mathbb{Z}$ and for every node $v \in V$ there is exactly one $i \in \mathbb{Z}$ such that $v = v_i$. From the previous construction, it follows that also the existence of a two-way Hamiltonian path in a given planar automatic graph of bounded degree is Σ^1_1-complete. Take the disjoint union of two copies of our main graph G and connect the two $\perp$-nodes with an edge. The resulting graph G' has a two-way Hamiltonian path iff G has a (one-way) Hamiltonian path. Moreover, since G is automatic and the class of automatic graphs is closed under disjoint unions, G' is automatic as well.

4 Remarks about large finite graphs

The main purpose of automatic presentations is the finite representation of infinite structures. But automatic presentations can be also used as a tool for the succinct representation of large finite structures. Note that a finite automaton with n states can accept a finite language with $2^{O(n)}$ elements, which may serve as the domain of a finite structure.

In general, given an automatic presentation (Γ, L, h) for a *finite* graph (V, E) together with an automaton A for the node set language L, it is clear that $|V|$ is bounded by $|\Gamma|^n$, where n is the number of states of A. It follows that for every graph problem L in NP, the succinct version of L, where the input graph is given by an automatic presentation, belongs to NEXPTIME. In particular, the Hamiltonian path problem belongs to NEXPTIME for this succinct input representation.

For the lower bound, consider for $n \geq 1$ the finite planar graph G_n that results from our main infinite graph G by restricting it to the graphs $G(k,\ell)$ for $k+\ell \leq n$ and the connecting XOR-graphs between these graphs. Then G_n has a Hamiltonian path if and only if the finite set of tiles $\mathscr{T}$ admits a tiling of the "triangle" $D_n = \{(k,\ell) \in \mathbb{N} \times \mathbb{N} \mid k+\ell \leq n\}$ (tilings of finite parts of the grid $\mathbb{N} \times \mathbb{N}$ are defined analogously to tilings of the whole grid). Now we can use a result of Fürer [8]: It is NEXPTIME-complete (under logspace reductions) to check for a given binary encoded number n and a finite set of tiles $\mathscr{T}$ whether $\mathscr{T}$ admits a tiling of D_n. Let us make a few remarks on Fürer's proof before continuing:

- Fürer proved NEXPTIME-completeness for tilings of the square $\{(k,\ell) \in \mathbb{N} \times \mathbb{N} \mid k,\ell \leq n\}$ instead of the triangle D_n. It is straightforward to adapt Fürer's proof for D_n.
- Fürer actually does not speak about NEXPTIME-completeness in his paper, but states explicit lower bounds. But in his proof he presents a generic reduction from the acceptance problem for nondeterministic exponential time Turing-machines to the problem of tiling $\{(k,\ell) \in \mathbb{N} \times \mathbb{N} \mid k,\ell \leq n\}$ for a given binary coded number.
- Fürer states that all his construction can be carried out in polynomial time, but it is straightforward to check that they can be carried out even in logspace.

Finally, it is easy to construct from a binary coded number n in logarithmic space an automatic presentation of the graph G_n. For this, we can basically use the automatic presentation of the infinite graph G, but restrict it to numbers of size at most n. Hence, we obtain:

Theorem 5. *It is NEXPTIME-complete under logspace reductions to check for a given automatic presentation of a finite planar graph, whether it has a Hamiltonian path.*

A variant of Thm. 5 was shown by Veith [28]. He considers finite structures that are represented by OBDDs (ordered binary decision diagrams). In this context, the node set of a graph is $\{0,1\}^n$ for some fixed n. The edge set is represented by an

OBDD over variables $x_1,\ldots,x_n,y_1,\ldots,y_n$. Here the tuple $(x_1,\ldots,x_n) \in \{0,1\}^n$ represents the initial vertex of an edge, whereas $(y_1,\ldots,y_n) \in \{0,1\}^n$ represents the final node. The variable order of the OBDDs in [28] is fixed to the interleaved order $x_1,y_1,x_2,y_2,\ldots,x_n,y_n$. Under this variable order, OBDDs exactly correspond to deterministic acyclic automata that work on the convolution $(x_1\cdots x_n)\otimes(y_1\cdots y_n)$.

In [28], the following upgrading theorem was shown (here, only formulated for the classes NP and NEXPTIME): If a graph problem L is NP-complete under quantifier free first-order reductions then $\mathrm{obdd}(L)$ (the class of all OBDDs of the above form that encode a graph from L) is NEXPTIME-complete under polynomial time reductions. Since the Hamiltonian path problem (HP) is NP-compete under quantifier free first-order reductions [26], it follows that obdd(HP) is NEXPTIME-complete under polynomial time reductions. Thm. 5 strengthens this result in two points: we obtain NEXPTIME-completeness (i) under logspace reductions and (ii) for *planar* graphs. It is not clear for us, whether the *planar* Hamiltonian path problem is still NP-complete under quantifier free first-order reductions.

5 Further graph problems

An *order tree* is a partial order $(A,\preceq)$ with a least element such that the set $\{a \in A \mid a \preceq b\}$ is finite and linearly ordered for every $b \in A$, a *successor tree* is the covering relation of an order tree. It is decidable, whether an automatic order tree has an infinite path [20]. The following result is in sharp contrast to this positive result.

Theorem 6. *It is Σ^1_1-complete to determine whether a given automatic successor tree T has an infinite path.*

The proof idea is to transform a recursive successor tree into an automatic one by adding the computation (i.e., sequence of transitions) that verifies the edge (u,v) as a path between the nodes u and v; a similar idea was used in [20, 15].

Let us now present some graph problems which are Σ^1_1-complete for recursive graphs, but decidable in automatic graphs. For this, we introduce, inspired by [18, 25], a fragment SO^r of second-order logic, which extends first-order logic with the infinity quantifier and modulo quantifiers. Every relation that is definable in first-order logic with the infinity quantifier and modulo quantifiers has a regular set of representatives [16, 5, 19]. We will extend this result to SO^r. The set of all formulas of SO^r is inductively defined as follows:

- Every atomic first-order formula is an SO^r-formula.
- $X(x_1,\ldots,x_k)$ for $x_1,\ldots,x_k$ first-order variables and X a k-ary second-order variable is an SO^r-formula.
- If φ and ψ are SO^r-formulas, then also $\varphi \vee \psi$ is an SO^r-formula.
- If φ is an SO^r-formula, then also $\neg\varphi$, $\exists x\,\varphi$, $\exists^\infty x\,\varphi$ (there are infinitely many x satisfying φ), $\exists^{(k,p)} x\,\varphi$ for $0 \le k < p \in \mathbb{N}$ (the number of x satisfying φ is finite and congruent k modulo p are SO^r-formulas.

- If φ is an $\mathrm{SO^r}$-formula and X is a second-order variable of arity k such that for every k-tuple of first-order variables $x_1,\dots,x_k$, φ contains the subformula $X(x_1,\dots,x_k)$ only negatively (i.e. within an odd number of negations), then also $\exists X$ infinite : φ is an $\mathrm{SO^r}$-formula.

Note that the restriction on φ in the last point means that if φ is satisfied for some k-ary relation $X = R$ and $Q \subseteq R$, then φ is also satisfied for $X = Q$.

Theorem 7. *From an automatic presentation (Γ, L, h) of an automatic structure $\mathscr{A}$ and an $\mathrm{SO^r}$-formula $\varphi(\bar{x})$ one can compute effectively an automaton for the convolution of the relation $\{(u_1,\dots,u_n) \in L^n \mid \mathscr{A} \models \varphi(h(u_1),\dots,h(u_n))\}$. Hence, if φ is an $\mathrm{SO^r}$-sentence, then $\mathscr{A} \models \varphi$ can be checked effectively.*

A variation of the proof of Thm. 7 yields the following result.

Theorem 8. *Let (Γ, L, h) be an automatic presentation of the structure $\mathscr{A}$ and let $\alpha(X)$ with X an n-ary relation variable be a formula of $\mathrm{SO^r}$ such that $\forall X, Y : \alpha(X \cup Y) \to \alpha(X)$ is a tautology and $\mathscr{A} \models \exists X$ infinite : α. Then one can construct $H \subseteq L^n$ regular such that $h(H)$ is infinite and $\mathscr{A} \models \alpha(h(H))$.*

We use Thm. 7 and 8 to show that two problems, which are Σ^1_1-complete for recursive structures [14], are decidable for automatic structures. First, by taking the $\mathrm{SO^r}$-formula $\exists X$ infinite $\forall x, y : (x, y \in X \Rightarrow (x, y) \in E)$, we get:

Corollary 1 (cf. [25, Thm. 3.20]). *It is decidable whether a given automatic graph contains an infinite clique. If an infinite clique exists, a regular set of representatives of an infinite clique can be computed.*

The second problem is the infinite version of maximal set cover considered by Hirst and Harel [14]. It asks whether, given a set $X = \{X_i \mid i \in \mathbb{N}\}$ of sets $X_i \subseteq \mathbb{N}$, there exists $A \subseteq \mathbb{N}$ with $\bigcup_{a \in A} X_a = \mathbb{N}$ and $\mathbb{N} \setminus A$ infinite. Note that the collection X can be represented as a set of pairs E with $(i, j) \in E$ iff $j \in X_i$. Then there exists A as required iff the directed graph $(\mathbb{N}, E)$ satisfies $\exists B$ infinite $\forall j \exists i : i \notin B \wedge (i, j) \in E$ (then A is the complement of B). Hence we get:

Corollary 2. *The infinite version of maximal set cover is decidable if the collection X is given as an automatic set of pairs. In case a set cover as required exists, an infinite such can be computed.*

6 Open problems

Hirst and Harel [14] gave an extensive list of problems that are Σ^1_1-complete in the recursive setting. Apart from the infinite version of the longest common subsequence problem, all of these problems are decidable or Σ^1_1-complete in the automatic setting (this follows easily from the problems considered in this paper). We are missing an explanation for this phenomenon – or a natural problem that is undecidable for automatic structures, but "simpler" than for recursive structures.

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