Automatic structures of bounded degree revisited

Dietrich Kuske and Markus Lohrey*

Universität Leipzig, Institut für Informatik, Germany

Abstract. It it shown that the first-order theory of an automatic structure, whose Gaifman graph has bounded degree, is decidable in doubly exponential space (for injective automatic presentations, this holds even uniformly). Presenting an automatic structure of bounded degree whose theory is hard for 2EXPSPACE, we also prove this result to be optimal. These findings close the gap left open in [14].

1 Introduction

The idea of an automatic structure goes back to Büchi and Elgot who used finite automata to decide, e.g., Presburger arithmetic [5]. Automaton decidable theories [9] and automatic groups [6] are similar concepts. A systematic study was initiated by Khoussainov and Nerode [10] who also coined the name "*automatic structure*". In essence, a structure is automatic if the elements of the universe can be represented as strings from a regular language (an element can be represented by several strings) and every relation of the structure can be recognized by a finite automaton with several heads that proceed synchronously. Automatic structures received increasing interest over the last years [3, 11, 12, 15, 1]. One of the main motivations for investigating automatic structures is that their (first-order) theories can be decided uniformly (i.e., the input is an automatic presentation and a first-order sentence). But even the theory of a specific automatic structure might be far from efficient: There exist automatic structures with a nonelementary theory. This motivates the search for subclasses of automatic structures with elementary theory. The first such class was identified by the second author in [14] who showed that the theory of every automatic structure of *bounded degree* can be decided in triply exponential alternating time with linearly many alternations. A structure has bounded degree, if in its Gaifman graph, the number of neighbors of a node is bounded by some fixed constant. The paper [14] also presents a specific automatic structure of bounded degree whose theory is hard for doubly exponential alternating time with linearly many alternations. Hence, an exponential gap between the upper and lower bound remained. An upper bound of 4-fold exponential alternating time with linearly many alternations was shown for *tree automatic structures* (which are defined analogously to automatic structures using tree automata) of bounded degree.

* The second author is supported by the DFG research project GELO.

Our paper [12] proves a triply exponential space bound for the theory of an injective ω-automatic structure (that is defined via Büchi-automata) of bounded degree; this result was recently applied to one-dimensional cellular automata [7]. Here, injectivity means that every element of the structure is represented by a *unique* ω-word from the underlying regular language.

In this paper, we achieve three goals: (i) We close the complexity gaps from [14] for automatic structures of bounded degree. (ii) We investigate, for the first time, the complexity of the *uniform* theory (where the automatic presentation is part of the input) of automatic structures of bounded degree. (iii) We refine our complexity analysis using the growth function of a structure. This function measures the size of a sphere in the Gaifman graph depending on the radius of the sphere. The growth function of a structure of bounded degree can be at most exponential.

Our main results are the following:

(a) The uniform theory for injective automatic presentations is 2EXPSPACE-complete. The lower bound already holds in the non-uniform setting, i.e. there exists an automatic structure of bounded degree with a 2EXPSPACE-complete theory.

(b) For every automatic structure of bounded degree, where the growth function is polynomially bounded, the theory is in EXPSPACE, and there exists an example with an EXPSPACE-complete theory.

In addition, the full version [13] of this extended abstract also contains analogous results for tree-automatic structures that had to be left out for space restrictions:

(c) The uniform theory for injective tree automatic presentations belongs to 4EXPTIME; the non-uniform one to 3EXPTIME for arbitrary tree automatic structures, and to 2EXPTIME if the growth function is polynomial. Our bounds for the non-uniform problem are sharp, i.e., there are tree automatic structures of bounded degree (and polynomial growth) with a 3EXPTIME-complete (2EXPTIME-complete, resp.) first-order theory.

We conclude this paper with some results on the complexity of first-order fragments with fixed quantifier alternation depth one or two on automatic structures of bounded degree. For a full version of this paper see [13].

2 Preliminaries

Let Γ be a finite alphabet and $w \in \Gamma^*$ be a finite word over Γ. The length of w is denoted by $|w|$. We also write $\Gamma^n = \{w \in \Gamma^* \mid n = |w|\}$.

Let $\exp(0, x) = x$ and $\exp(n+1, x) = 2^{\exp(n,x)}$ for $x \in \mathbb{N}$. We assume basic knowledge in complexity theory. For $k \geq 1$, we denote with kEXPSPACE (resp. kEXPTIME) the class of all problems that can be accepted in space (resp. time) $\exp(k, n^{O(1)})$ on a deterministic Turing machine. For 1EXPSPACE we write just EXPSPACE, EXPTIME is to be understood similarly. A problem is called *elementary* if it belongs to kEXPTIME for some $k \in \mathbb{N}$.

Recall that emptiness and inclusion of the languages of finite nondeterministic automata are complete for NL (nondeterministic logspace) and PSPACE (polynomial space), resp..

2.1 Structures and first-order logic

A *signature* is a finite set $\mathcal{S}$ of relational symbols, where every symbol $r \in \mathcal{S}$ has some fixed arity m_r. The notion of an $\mathcal{S}$-structure (or model) is defined as usual in logic. We only consider relational structures. Sometimes, we will also use constants, but in our context, a constant c can be replaced by the unary relation $\{c\}$. Let us fix an $\mathcal{S}$-structure $\mathcal{A} = (A, (r^{\mathcal{A}})_{r \in \mathcal{S}})$, where $r^{\mathcal{A}} \subseteq A^{m_r}$. To simplify notation, we will write $a \in \mathcal{A}$ for $a \in A$. For $B \subseteq A$ we define the restriction $\mathcal{A}{\restriction}B = (B, (r^{\mathcal{A}} \cap B^{m_r})_{r \in \mathcal{S}})$. Given further constants $a_1, \ldots, a_k \in \mathcal{A}$, we write $(\mathcal{A}, a_1, \ldots, a_k)$ for the structure $(A, (r^{\mathcal{A}})_{r \in \mathcal{S}}, a_1, \ldots, a_k)$. In the rest of the paper, we will always identify a symbol $r \in \mathcal{S}$ with its interpretation $r^{\mathcal{A}}$. A *congruence* on the structure $\mathcal{A} = (A, (r)_{r \in \mathcal{S}})$ is an equivalence relation $\equiv$ on A such that for every $r \in \mathcal{S}$ and all $a_1, b_1, \ldots, a_{m_r}, b_{m_r} \in A$ we have: If $(a_1, \ldots, a_{m_r}) \in r$ and $a_1 \equiv b_1, \ldots, a_{m_r} \equiv b_{m_r}$, then also $(b_1, \ldots, b_{m_r}) \in r$. As usual, the equivalence class of $a \in A$ w.r.t. $\equiv$ is denoted by $[a]_\equiv$ or just $[a]$ and $A/_\equiv$ denotes the set of all equivalence classes. We define the *quotient structure* $\mathcal{A}/_\equiv = (A/_\equiv, (r/_\equiv)_{r \in \mathcal{S}})$, where $r/_\equiv = \{([a_1], \ldots, [a_{m_r}]) \mid (a_1, \ldots, a_{m_r}) \in r\}$.

The *Gaifman-graph* $G(\mathcal{A})$ of the $\mathcal{S}$-structure $\mathcal{A}$ is the symmetric graph on the universe A of $\mathcal{A}$, which contains an edge between a and b if and only if there exists a tuple $(a_1, \ldots, a_{m_r}) \in r$ in some of the relations $r \in \mathcal{S}$ such that a and b both belong to $\{a_1, \ldots, a_{m_r}\}$. With $d_{\mathcal{A}}(a, b)$, where $a, b \in \mathcal{A}$, we denote the distance between a and b in $G(\mathcal{A})$, i.e., it is the length of a shortest path connecting a and b in $G(\mathcal{A})$. For $a \in \mathcal{A}$ and $d \geq 0$ we denote with $S_{\mathcal{A}}(d, a) = \{b \in A \mid d_{\mathcal{A}}(a, b) \leq d\}$ the d-sphere around a. If $\mathcal{A}$ is clear from the context, then we will omit the subscript $\mathcal{A}$. We say that the structure $\mathcal{A}$ is *locally finite* if its Gaifman graph $G(\mathcal{A})$ is locally finite (i.e., every node has finitely many neighbors). Similarly, the structure $\mathcal{A}$ has *bounded degree*, if $G(\mathcal{A})$ has bounded degree, i.e., there exists a constant δ such that every $a \in A$ is adjacent to at most δ many other nodes in $G(\mathcal{A})$; the minimal such δ is called the *degree of* $\mathcal{A}$. For a structure $\mathcal{A}$ of bounded degree we define its *growth function* $g_{\mathcal{A}} : \mathbb{N} \to \mathbb{N}$ as $g_{\mathcal{A}}(n) = \max\{|S_{\mathcal{A}}(n, a)| \mid a \in \mathcal{A}\}$. Note that if the function $g_{\mathcal{A}}$ is not bounded then $g_{\mathcal{A}}(n) \geq n$ for all $n \geq 1$. For us, it is more convenient to not have a bounded function describing the growth. Therefore, we define the *normalized growth function* $g'_{\mathcal{A}}$ by $g'_{\mathcal{A}}(n) = \max\{n, g_{\mathcal{A}}(n)\}$. Note that $g_{\mathcal{A}}$ and $g'_{\mathcal{A}}$ are different only in the case that all connected components of $\mathcal{A}$ contain at most m elements (for some fixed m). Clearly, $g'_{\mathcal{A}}(n)$ can grow at most exponentially if $\mathcal{A}$ has bounded degree. We say that $\mathcal{A}$ has *exponential growth* if $g'_{\mathcal{A}}(n) \in 2^{\Omega(n)}$. If $g'_{\mathcal{A}}(n) \in n^{O(1)}$, then $\mathcal{A}$ has *polynomial growth*.

We consider (first-order) formulas with equality over the signature $\mathcal{S}$. The *quantifier depth* of a formula φ is the maximal nesting of quantifiers in φ. A formula without free variables is called *closed*. The *theory* of $\mathcal{A}$, denoted by $\mathrm{Th}(\mathcal{A})$, is the set of all closed formulas φ with $\mathcal{A} \models \varphi$.

2.2 Structures from automata

Automatic structures Next we introduce automatic structures, more details can be found in [10, 3]. Let us fix $n \in \mathbb{N}$ and a finite alphabet Γ. Let $\$ \notin \Gamma$ be an additional padding symbol. For words $w_1, w_2, \ldots, w_n \in \Gamma^*$ we define the *convolution* $w_1 \otimes w_2 \otimes \cdots \otimes w_n$, which is a word over the alphabet $(\Gamma \cup \{\$\})^n$, as follows: Let $w_i = a_{i,1} a_{i,2} \cdots a_{i,k_i}$ with $a_{i,j} \in \Gamma$ and $k = \max\{k_1, \ldots, k_n\}$. For $k_i < j \leq k$ define $a_{i,j} = \$$. Then $w_1 \otimes \cdots \otimes w_n = (a_{1,1}, \ldots, a_{n,1}) \cdots (a_{1,k}, \ldots, a_{n,k})$. Thus, for instance $aba \otimes bbabb = (a,b)(b,b)(a,a)(\$,b)(\$,b)$. An n-ary relation $R \subseteq (\Gamma^*)^n$ is called *automatic* if the language $\{w_1 \otimes \cdots \otimes w_n \mid (w_1, \ldots, w_n) \in R\}$ is a regular language.

An *m-dimensional (synchronous) automaton* over Γ is just a finite automaton $A = (Q, \Delta, q_0, F)$ over $(\Gamma \cup \{\$\})^m$ such that $L(A) \subseteq \{w_1 \otimes \cdots \otimes w_n \mid w_1, \ldots, w_m \in \Gamma^*\}$. Such an automaton defines an m-ary relation

$$R(A) = \{(w_1, \ldots, w_m) \mid w_1 \otimes \cdots \otimes w_m \in L(A)\} .$$

We define the size $|A|$ of A as $\max\{1, |\Delta|\} \cdot m$. Reasonably assuming that every state is the target state of some transition and that every letter from Γ appears in some transition (implying $|\Gamma|, |Q| \leq |\Delta|$) , the size $|A|$ bounds the number of bits needed to store A (up to some polynomial).

An *automatic presentation* is a tuple $P = (\Gamma, \mathcal{S}, A_0, A_=, (A_r)_{r \in \mathcal{S}})$, where: (i) Γ is a finite alphabet, (ii) $\mathcal{S}$ is the signature of P (as before m_r is the arity of the symbol $r \in \mathcal{S}$), (iii) A_0 is an automaton over the alphabet Γ, (iv) for every $r \in \mathcal{S}$, A_r is an m_r-dimensional automaton over Γ with $R(A_r) \subseteq L(A_0)^{m_r}$, and (v) $A_=$ is a 2-dimensional automaton over Γ such that $R(A_=) \subseteq L(A_0)^2$ is a congruence on the structure $(L(A_0), (R(A_r))_{r \in \mathcal{S}})$. This presentation P is *injective* if $R(A_=)$ is the identity relation on $L(A_0)$. The structure presented by P is the quotient $\mathcal{A}(P) = (L(A_0), (R(A_r))_{r \in \mathcal{S}})/_{R(A_=)}$. A structure $\mathcal{A}$ is *automatic* if there exists an automatic presentation P such that $\mathcal{A} \simeq \mathcal{A}(P)$. We will write $[u]$ for the element $[u]_{R(A_=)}$ ($u \in L(A_0)$) of the structure $\mathcal{A}(P)$. The presentation P has *bounded degree* if the structure $\mathcal{A}(P)$ has bounded degree. The size of the presentation $P = (\Gamma, \mathcal{S}, A_0, A_=, (A_r)_{r \in \mathcal{S}})$ is $|P| = |A_0| + |A_=| + \sum_{r \in \mathcal{S}} |A_r|$. Note that $|\mathcal{S}| \leq |P|$ and $m_r \leq |P|$ for all $r \in \mathcal{S}$.

Typical examples of automatic structures are $(\mathbb{N}, +)$ and $(\mathbb{Q}, \leq)$. Examples of automatic structures of bounded degree are transition graphs of Turing machines and Cayley-graphs of automatic groups [6] as well as the queue structure [16] (the set of finite words together with functions prefixing and suffixing a word by a fixed letter). There are automatic structures $\mathcal{A}$ of bounded degree with growth $g_{\mathcal{A}}(n) \in 2^{\Theta(\sqrt{n})}$ [13].

We will consider the following classes of automatic presentations[1]:

- SA: the class of all automatic presentations.
- SAb: the class of all automatic presentations of bounded degree.
- iSAb: the class of all injective automatic presentations of bounded degree.

[1] The letter S in the below classes refers to “string”, the full paper [13] also contains classes starting with T that refers to “tree”

The model checking problem For the above classes of automatic presentations, we will be interested in the following decision problems.

Definition 2.1. *Let* C *be a class of automatic presentations. Then the* model checking problem $\mathrm{MC}(\mathsf{C})$ for C *denotes the set of all pairs* (P, φ) *where* $P \in \mathsf{C}$, *and* φ *is a closed formula over the signature of* P *such that* $\mathcal{A}(P) \models \varphi$.

If $\mathsf{C} = \{P\}$ is a singleton, then the model checking problem $\mathrm{MC}(\mathsf{C})$ for C can be identified with the theory of the structure $\mathcal{A}(P)$. An algorithm deciding the model checking problem for a class C decides the theories of each element of C uniformly.

The following two results are the main motivations for investigating automatic structures.

Proposition 2.2 (cf. [10]). *There is an algorithm that computes, from an automatic presentation* $P = (\Gamma, \mathcal{S}, A_0, A_=, (A_r)_{r\in\mathcal{S}})$ *and a formula* $\varphi(x_1, \ldots, x_m)$, *an* m*-dimensional automaton* A *over* Γ *with* $R(A) = \{(u_1, \ldots, u_m) \in L(A_0)^m \mid \mathcal{A}(P) \models \varphi([u_1], \ldots, [u_m])\}$.

The automaton is constructed by induction on the structure of the formula φ: disjunction corresponds to the disjoint union of automata, existential quantification to projection, and negation to complementation. The following result is a direct consequence.

Theorem 2.3 (cf. [10]). *The model checking problem* $\mathrm{MC}(\mathsf{SA})$ *for all automatic presentations is decidable. In particular, the theory* $\mathrm{Th}(\mathcal{A})$ *of every automatic structure* $\mathcal{A}$ *is decidable.*

Strictly speaking, [10] devices algorithms that, given an automatic presentation and a closed formula, decide whether the formula holds in the presented structure. But a priori, it is not clear whether it is decidable, whether a given tuple $(\Gamma, \mathcal{S}, A_0, A_=, (A_r)_{r\in\mathcal{S}})$ is an automatic presentation. Prop. 2.5(a) below shows that SA is indeed decidable in polynomial space, which then completes the proof of this theorem.

Thm. 2.3 holds even if we add quantifiers for "there are infinitely many x such that $\varphi(x)$" [2, 3] and "the number of elements satisfying $\varphi(x)$ is divisible by k" (for $k \in \mathbb{N}$) [11]. This implies in particular that it is decidable whether an automatic presentation describes a locally finite structure. But the decidability of the theory is far from efficient, since there are automatic structures with a nonelementary first-order theory [3]. An example for this is the infinite binary tree with the prefix relation, see e.g. [4, Example 8.3]. A locally finite example can be obtained by taking the disjoint union of all finite binary-labeled linear orders, see e.g. [4].

Preliminary complexity results It will be convenient to work with injective automatic presentations. The following lemma says that this is no restriction, if we allow an exponential jump in complexity.

Lemma 2.4 ([10, Cor. 4.3]). *From $P \in \mathsf{SA}$ we can compute in time $2^{O(|P|)}$ an injective automatic presentation $P' \in \mathsf{iSA}$ with $\mathcal{A}(P) \simeq \mathcal{A}(P')$.*

Next, we give complexity bounds for the class of all automatic structures as well as for those of bounded degree.

Proposition 2.5. *(a) The class* SA *is* PSPACE*-complete and (b) the class* SAb *belongs to* $\mathsf{EXPTIME}$*.*

Proof. Statement (a) is shown in the full version [13]. For (b) we can assume by (a) that the input indeed belongs to SA (which can be checked in polynomial space and therefore in exponential time). In exponential time, the automatic presentation can then be transformed into an equivalent injective one $P \in \mathsf{iSA}$ of exponential size. Using simple automata constructions, we can compute a 2-dimensional automaton A for the edge relation of the Gaifman-graph of $\mathcal{A}(P)$ (in fact, A can be computed in time polynomial in $|P|$). Since P is injective (i.e. every equivalence class $[u]$ is the singleton $\{u\}$), $\mathcal{A}(P)$ is of bounded degree iff A (seen as a transducer) is finite-valued. But this is decidable in time polynomial in $|P|$ [17]. Since P is exponential in the input, this completes the proof. □

In contrast to this decidability results, it is undecidable, whether a given automatic structure of bounded degree has polynomial growth, see the complete version [13].

Finally, since we deal with structures of bounded degree, it will be important to estimate the degree of such a structure given its presentation. Such estimates are provided by the following result.

Proposition 2.6. *The following holds:*

(a) If $P \in \mathsf{iSAb}$, then the degree of $\mathcal{A}(P)$ is bounded by $\exp(1, |P|^{O(1)})$.
(b) If $P \in \mathsf{SAb}$, then the degree of $\mathcal{A}(P)$ is bounded by $\exp(2, |P|^{O(1)})$.

Proof. For statement (a) let $P \in \mathsf{iSAb}$. From P we can construct a 2-dimensional automaton A of size $|P|^{O(1)}$ that accepts the edge relation of the Gaifman graph of $\mathcal{A}(P)$. Then the degree of $\mathcal{A}(P)$ equals the maximal out-degree of the relation $R(A)$. For string transducers, this number is exponential in the size of A, i.e., it is in $\exp(1, |P|^{O(1)})$ [17].

For $P \in \mathsf{SAb}$, the bound $\exp(2, |P|^{O(1)})$ follows immediately from Lemma 2.4 and (a). □

The bound in Prop. 2.6 for $P \in \mathsf{iSAb}$ is sharp, see the complete version [13] for an example.

3 Upper bounds

It is the aim of this section to give an algorithm that decides the theory of an automatic structure of bounded degree. The algorithm from Thm. 2.3 (that in particular solves this problem) is based on Prop. 2.2, i.e., the inductive construction of an automaton accepting all satisfying assignments. Differently, we base our algorithm on Gaifman's Thm. 3.1, i.e., on the combinatorics of spheres. We therefore start with some model theory.

3.1 Model-theoretic background

For a structure $\mathcal{A}$, $\overline{a} = (a_1, \ldots, a_k) \in \mathcal{A}^k$ and $d \geq k \geq 0$, we denote with $\mathcal{A}[d, \overline{a}]$ the induced substructure $\mathcal{A}\restriction \bigcup_{i=1}^{k} S(7^{d-i}, a_i)$. The following locality principle of Gaifman implies that super-exponential distances cannot be handled in first-order logic:

Theorem 3.1 ([8]). *Let $\mathcal{A}$ be a structure, $\overline{a}, \overline{b} \in \mathcal{A}^k$ and $d \geq 0$ such that $(\mathcal{A}[d+k, \overline{a}],\ \overline{a}) \simeq (\mathcal{A}[d+k, \overline{b}],\ \overline{b})$ (i.e. there is an isomorphism between the two induced substructures $\mathcal{A}[d+k, \overline{a}]$ and $\mathcal{A}[d+k, \overline{b}]$ that maps the i^{th} component of $\overline{a}$ to the i^{th} component of $\overline{b}$ for all $1 \leq i \leq k$). Then, for every formula $\varphi(x_1, \ldots, x_k)$ of quantifier depth at most d, we have: $\mathcal{A} \models \varphi(\overline{a}) \iff \mathcal{A} \models \varphi(\overline{b})$.*

Let $\mathcal{S}$ be a signature and let $k, d \in \mathbb{N}$ with $0 \leq k \leq d$. A *potential (d,k)-sphere* is a tuple $(\mathcal{B}, \overline{b})$ such that $\mathcal{B}$ is an $\mathcal{S}$-structure, $\overline{b} \in \mathcal{B}^k$, and $\mathcal{B} = \mathcal{B}[d, \overline{b}]$. There is only one potential $(d, 0)$-sphere namely the empty sphere $\emptyset$. For our later applications, $\mathcal{B}$ will be always a finite structure, but in this subsection finiteness is not needed. The potential (d,k)-sphere $(\mathcal{B}, \overline{b})$ is *realized in the structure $\mathcal{A}$* if there exists $\overline{a} \in \mathcal{A}^k$ such that $(\mathcal{A}[d, \overline{a}], \overline{a}) \simeq (\mathcal{B}, \overline{b})$.

Let $\sigma = (\mathcal{B}, \overline{b})$ be a potential (d,k)-sphere and let $\sigma' = (\mathcal{C}, \overline{c}, c)$ be a potential $(d, k+1)$-sphere ($k+1 \leq d$, $\overline{c} \in \mathcal{C}^k$, $c \in \mathcal{C}$). Then σ' *extends* σ (abbreviated $\sigma \preceq \sigma'$) if $\sigma \simeq (\mathcal{C}[d, \overline{c}], \overline{c})$. The following definition is the basis for our decision procedure.

Definition 3.2. *Let $\mathcal{A}$ be an $\mathcal{S}$-structure, $\psi(y_1, \ldots, y_k)$ a formula of quantifier depth at most d, and let $\sigma = (\mathcal{B}, \overline{b})$ be a potential $(d+k, k)$-sphere. The Boolean value $\psi_\sigma \in \{0, 1\}$ is defined inductively as follows:*

- *If $\psi(y_1, \ldots, y_k)$ is an atomic formula, then*

$$\psi_\sigma = 1 \iff \mathcal{B} \models \psi(\overline{b}). \tag{1}$$

- $(\neg\theta)_\sigma = 1 - \theta_\sigma$ *and* $(\alpha \vee \beta)_\sigma = \max\{\alpha_\sigma, \beta_\sigma\}$
- *If $\psi(y_1, \ldots, y_k) = \exists y_{k+1} \theta(y_1, \ldots, y_k, y_{k+1})$ then*

$$\psi_\sigma = \max\{\theta_{\sigma'} \mid \sigma' \text{ is a potential}(d+k, k+1)\text{-sphere with } \sigma \preceq \sigma' \text{ that is realized in } \mathcal{A}\}\ . \tag{2}$$

The following result ensures for every closed formula ψ that $\psi_\emptyset = 1$ if and only if $\mathcal{A} \models \psi$. Hence the above definition can possibly be used to decide validity of the formula φ in the structure $\mathcal{A}$.

Proposition 3.3. *Let $\mathcal{S}$ be a signature, $\mathcal{A}$ an $\mathcal{S}$-structure, $\overline{a} \in \mathcal{A}^k$, $\psi(y_1, \ldots, y_k)$ a formula of quantifier depth at most d, and $\sigma = (\mathcal{B}, \overline{b})$ a potential $(d+k, k)$-sphere with*

$$(\mathcal{A}[d+k, \overline{a}], \overline{a}) \simeq \sigma\ . \tag{3}$$

Then $\mathcal{A} \models \psi(\overline{a}) \iff \psi_\sigma = 1$.

Proof. We prove the lemma by induction on the structure of ψ. First assume that ψ is atomic, i.e. $d = 0$. We have

$$\psi_\sigma = 1 \overset{(1)}{\iff} \mathcal{B} \models \psi(\bar{b}) \overset{(3)}{\iff} \mathcal{A}[0+k, \overline{a}] \models \psi(\overline{a}) \iff \mathcal{A} \models \psi(\overline{a}) ,$$

where the last equivalence holds since ψ is atomic. The cases $\psi = \neg\theta$ and $\psi = \alpha \vee \beta$ are straightforward.

We finally consider the case $\psi(y_1, \dots, y_k) = \exists y_{k+1} \theta(y_1, \dots, y_k, y_{k+1})$. First assume that $\psi_\sigma = 1$. By (2), some potential $(d+k, k+1)$-sphere σ' is realized in $\mathcal{A}$ with $\sigma \preceq \sigma'$ and $\theta_{\sigma'} = 1$. Since σ' is realized, there exist $\overline{a}' \in \mathcal{A}^k$, $a' \in \mathcal{A}$ with

$$(\mathcal{A}[d+k, \overline{a}', a'], \overline{a}', a') \simeq (\mathcal{B}', \bar{b}, b) = \sigma' \ . \tag{4}$$

By induction, we have $\mathcal{A} \models \theta(\overline{a}', a')$ and therefore $\mathcal{A} \models \psi(\overline{a}')$. From (4), $\sigma \preceq \sigma'$, and (3), we also obtain

$$(\mathcal{A}[d+k, \overline{a}'], \overline{a}') \simeq (\mathcal{A}[d+k, \overline{a}], \overline{a})$$

and therefore by Gaifman's Thm. 3.1 $\mathcal{A} \models \psi(\overline{a})$.

Conversely, let $a \in \mathcal{A}$ with $\mathcal{A} \models \theta(\overline{a}, a)$. Let $\sigma' = (\mathcal{B}', \bar{b}, b)$ be the unique (up to isomorphism) potential $(d+k, k+1)$-sphere such that

$$(\mathcal{A}[d+k, \overline{a}, a], \overline{a}, a) \simeq (\mathcal{B}', \bar{b}, b) \ . \tag{5}$$

Then (3) implies $\sigma \preceq \sigma'$. Moreover, by (5), σ' is realized in $\mathcal{A}$, and $\mathcal{A} \models \theta(\overline{a}, a)$ implies by induction $\theta_{\sigma'} = 1$. Hence, by (2), we get $\psi_\sigma = 1$ which finishes the proof. ◻

3.2 The decision procedure

Now suppose we want to decide whether the closed formula φ holds in an automatic structure $\mathcal{A}$ of *bounded degree*. By Prop. 3.3 it suffices to compute the Boolean value $\varphi_\emptyset$. This computation will follow the inductive definition of φ_σ from Def. 3.2. Since every (d,k)-sphere that is realized in $\mathcal{A}$ is finite, we only have to deal with finite spheres. The crucial part of our algorithm is to determine whether a finite potential (d,k)-sphere is realized in $\mathcal{A}$. In the following, for a finite potential (d,k)-sphere $\sigma = (\mathcal{B}, b_1, \dots, b_k)$, we denote with $|\sigma|$ the number of elements and with $\delta(\sigma)$ the degree of the finite structure $\mathcal{B}$.

For a class of automatic presentations C the *realizability problem* REAL(C) *for* C denotes the set of all pairs (P, σ) where $P \in \mathsf{C}$ and σ is a *finite* potential (d,k)-sphere over the signature of P for some $0 \le k \le d$ such that σ can be realized in $\mathcal{A}(P)$. In the complexity estimates in the following lemma, σ is the input potential (d,k)-sphere and P is the input automatic presentation.

Lemma 3.4. REAL(iSA) *can be solved in space* $|\sigma|^{O(|P|)} \cdot 2^{O(\delta(\sigma))}$.

Proof. Let $P = (\Gamma, \mathcal{S}, A_0, A_=, (A_r)_{r \in \mathcal{S}}) \in \mathsf{iSA}$. Let $\sigma = (\mathcal{B}, b_1, \dots, b_k)$ and let $c_1, \dots, c_{|\sigma|}$ be a list of all elements of $\mathcal{B}$; every b_i occurs in this list. Let $E_{\mathcal{A}(P)}$ (resp. $E_{\mathcal{B}}$) be the edge relation of the Gaifman graph $G(\mathcal{A}(P))$ (resp. $G(\mathcal{B})$). Then σ is realized in $\mathcal{A}(P)$ iff there are $u_1, \dots, u_{|\sigma|} \in \Gamma^*$ with

(a) $u_i \in L(A_0)$ for all $1 \leq i \leq |\sigma|$,
(b) $u_i \neq u_j$ for all $1 \leq i < j \leq |\sigma|$,
(c) For all $r \in \mathcal{S}$: $(u_{i_1}, \ldots, u_{i_{m_r}}) \in R(A_r)$ if and only if $(c_{i_1}, \ldots, c_{i_{m_r}}) \in r^{\mathcal{B}}$, and
(d) there is no $v \in L(A_0)$ such that, for some $1 \leq j \leq |\sigma|$ and $1 \leq i \leq k$ with $d(c_j, b_i) < 2^{d-i}$, we have: $(u_j, v) \in E_{\mathcal{A}(P)}$ and $v \notin \{u_p \mid (c_j, c_p) \in E_{\mathcal{B}}\}$.

Then (a-c) express that the mapping $c_i \mapsto u_i$ is well-defined and an embedding of $\mathcal{B}$ into $\mathcal{A}(P)$. In (d), $(u_j, v) \in E_{\mathcal{A}(P)}$ implies that v belongs to $\bigcup_{1 \leq i \leq k} S(2^{d-i}, u_i)$. Hence (d) expresses that all elements of $\bigcup_{1 \leq i \leq k} S(2^{d-i}, u_i)$ belong to the image of this embedding.

Using standard automata constructions for Boolean operations and projection, we can construct a $|\sigma|$-dimensional automaton A over the alphabet Γ that checks (a-d). A detailed size estimate shows that A has at most $\exp(1, |\sigma|^{O(|P|)} \cdot 2^{O(\delta(\sigma))})$ many states. Hence checking emptiness of its language (and therefore realizability of σ in $\mathcal{A}(P)$) can be done in space logarithmic to the number of states, i.e., in space $|\sigma|^{O(|P|)} \cdot 2^{O(\delta(\sigma))}$ which proves the statement. □

In the following, for an automatic presentation P of bounded degree, g'_P denotes the normalized growth function $g'_{\mathcal{A}(P)}$ of the structure $\mathcal{A}(P)$. In the complexity estimates in the following theorem, φ is the input sentence and P is the input automatic presentation.

Theorem 3.5. MC(iSAb) *can be solved in space* $g'_P(2^{|\varphi|})^{O(|P|)} \exp(2, |P|^{O(1)})$.

Proof. It suffices by Prop. 3.3 to compute the Boolean value $\varphi_\emptyset$. Recall the inductive definition of φ_σ from Def. 3.2 that we now translated into an algorithm for computing $\varphi_\emptyset$. Such an algorithm has to handle potential (d, k)-spheres for $1 \leq k \leq d \leq |\varphi|$ (d is the quantifier rank of φ) that are realized in $\mathcal{A}(P)$. The number of nodes of a potential (d, k)-sphere realized in $\mathcal{A}(P)$ is bounded by $k \cdot g'_P(2^d) \leq g'_P(2^{|\varphi|})^{O(1)}$ since $k < 2^{|\varphi|} \leq g'_P(2^{|\varphi|})$. The number of relations of $\mathcal{A}(P)$ as well as each arity is bounded by $|P|$. Hence, any potential (d, k)-sphere can be stored in space $|P| \cdot g'_P(2^{|\varphi|})^{O(|P|)} = g'_P(2^{|\varphi|})^{O(|P|)}$.

The set of (d, k)-spheres with $0 \leq k \leq d$ (ordered by the extension relation $\preceq$) forms a tree of depth $d+1$. The algorithm visits the nodes of this tree in a depth-first manner and descents when unraveling an existential quantifier. Hence, we have to store $d + 1 \leq |\varphi|$ many spheres, for which space $|\varphi| \cdot g'_P(2^{|\varphi|})^{O(|P|)} = g'_P(2^{|\varphi|})^{O(|P|)}$ is sufficient.

Moreover, during the unraveling of a quantifier, the algorithm has to check realizability of a potential (d, k)-sphere for $1 \leq k \leq d$. Any such sphere has at most $g'_P(2^{|\varphi|})^{O(1)}$ many elements and the degree δ of $\mathcal{A}$ is bounded by $\exp(1, |P|^{O(1)})$ by Prop. 2.6(a). Hence, by Lemma 3.4, realizability can be checked in space $g'_P(2^{|\varphi|})^{O(|P|)} \cdot \exp(2, |P|^{O(1)})$.

At the end, we have to check whether a tuple $\overline{b}$ satisfies an atomic formula $\psi(\overline{y})$, which is trivial. Thus, the totally needed space is at most $g'_P(2^{|\varphi|})^{O(|P|)} \cdot \exp(2, |P|^{O(1)})$. □

We derive a number of consequences on the combined and expression complexity of automatic structures of bounded degree. The first one concerns the combined complexity and is a direct consequence of Thm. 3.5:

Corollary 3.6. *The following holds:*

(a) MC(iSAb) *is in* 2EXPSPACE*.*
(b) MC(SAb) *is in* 3EXPSPACE*.*

Proof. Statement (a) follows from Thm. 3.5 and the fact that (i) $g'_{\mathcal{A}}(2^{|\varphi|}) \leq \delta^{2^{|\varphi|}}$ if δ is the degree of $\mathcal{A}(P)$ and (ii) Prop. 2.6(a), which allows to bound δ by $2^{|P|^{O(1)}}$. Statement (b) follows from (a) and Lemma 2.4, which allows to make an automatic presentation injective with an exponential blow up. □

Next we concentrate on the expression complexity, i.e., we fix the structure.

Corollary 3.7. *If $\mathcal{A}$ is an automatic structure of bounded degree, then* Th($\mathcal{A}$) *belongs to* 2EXPSPACE*. If in addition, $\mathcal{A}$ has also polynomial growth, then* Th($\mathcal{A}$) *belongs to* EXPSPACE*.*

Proof. Since $\mathcal{A}$ is automatic, it has a fixed injective automatic presentation P, i.e., $|P|$ is a fixed constant. Hence, the first statement follows immediately from Thm. 3.5. If $\mathcal{A}$ has in addition polynomial growth, then, again, the claim follows immediately from Thm. 3.5 since $g'_{\mathcal{A}}(2^{|\varphi|})^{O(|P|)} = 2^{O(|\varphi|)}$. □

4 Lower bounds

In this section, we will prove that the upper bounds for the expression complexities (Cor. 3.7) are sharp. This will imply that the upper bounds for the combined complexity for injective automatic presentations from Thm. 3.5 is sharp as well.

For a binary relation r and $m \in \mathbb{N}$ we denote with r^m the m-fold composition of r. The following lemma is folklore.

Lemma 4.1. *Let the signature $\mathcal{S}$ contain a binary symbol r. From a given number m (encoded unary), we can construct in linear time a formula $\varphi_m(x, y)$ such that for every $\mathcal{S}$-structure $\mathcal{A}$ and all elements $a, b \in \mathcal{A}$ we have: $(a, b) \in r^{2^m}$ if and only if $\mathcal{A} \models \varphi_m(a, b)$.*

For a bit string $u = a_1 \cdots a_m$ ($a_i \in \{0, 1\}$) let $\mathrm{val}(u) = \sum_{i=0}^{m-1} a_{i+1} 2^i$ be the integer value represented by u. Vice versa, for $0 \leq i < 2^m$ let $\mathrm{bin}_m(i) \in \{0, 1\}^m$ be the unique string with $\mathrm{val}(\mathrm{bin}_m(i)) = i$.

Theorem 4.2. *There exists a fixed automatic structure $\mathcal{A}$ of bounded degree such that* Th($\mathcal{A}$) *is* 2EXPSPACE*-hard.*

Proof. Let M be a fixed Turing machine with a space bound of $\exp(2, n)$ such that M accepts a 2EXPSPACE-complete language; such a machine exists by standard arguments. Let Γ be the tape alphabet, $\Sigma \subseteq \Gamma$ be the input alphabet, and Q be the set of states. The initial (resp. accepting) state is $q_0 \in Q$ (resp. $q_f \in Q$), the blank symbol is $\Box \in \Gamma \setminus \Sigma$. Let $\Omega = Q \cup \Gamma$. A configuration of M is described by a string from $\Gamma^* Q \Gamma^+ \subseteq \Omega^+$ (later, symbols of configurations will be preceded with additional counters). For two configurations u and v with $|u| = |v|$ we write $u \vdash_M v$ if u can evolve with a single M-transition into v. Note that there exists a relation $\alpha_M \subseteq \Omega^3 \times \Omega$ such that for all configurations $u = a_1 \cdots a_m$ and $v = b_1 \cdots b_m$ ($a_i, b_i \in \Omega$) we have $u \vdash_M v$ if and only if

$$\forall i \in \{2, \dots, m-1\} : (a_{i-1} a_i a_{i+1}, b_i) \in \alpha_M. \tag{6}$$

Let $\Delta = \{0, 1, \#\} \cup \Omega$, and let $\pi : \Delta \to \Omega \cup \{\#\}$ be the projection morphism with $\pi(a) = a$ for $a \in \Omega \cup \{\#\}$ and $\pi(0) = \pi(1) = \varepsilon$. For $m \in \mathbb{N}$, a string $x \in \Delta^*$ is an *accepting* 2^m*-computation* if x can be factorized as $x = x_1 \# x_2 \# \cdots x_n \#$ for some $n \geq 1$ such that:

- For every $1 \leq i \leq n$ there exist $a_{i,0}, \dots, a_{i,2^m-1} \in \Omega$ such that $x_i = \prod_{j=0}^{2^m-1} \mathrm{bin}_m(j) a_{i,j}$.
- For every $1 \leq i \leq n$, $\pi(x_i) \in \Gamma^* Q \Gamma^+$.
- $\pi(x_1) \in q_0 \Sigma^* \Box^*$ and $\pi(x_n) \in \Gamma^* q_f \Gamma^+$
- For every $1 \leq i < n$, $\pi(x_i) \vdash_M \pi(x_{i+1})$.

From M we now construct a fixed automatic structure $\mathcal{A}$ of bounded degree. We start with the following regular language U_0:

$$\begin{aligned} U_0 \ &= \pi^{-1}((\Gamma^* Q \Gamma^+ \#)^*) \ \cap && (7)\\ &\quad (0^+ \Omega(\{0,1\}^+ \Omega)^* 1^+ \Omega \#)^+ \ \cap && (8)\\ &\quad 0^+ q_0 (\{0,1\}^+ \Sigma)^* (\{0,1\}^+ \Box)^* \# \Delta^* \ \cap && (9)\\ &\quad \Delta^* q_f (\Delta \setminus \{\#\})^* \# && (10) \end{aligned}$$

A string $x \in U_0$ is a candidate for an accepting 2^m-computation of M. With (7) we describe the basic structure of such a computation, it consists of a list of configurations separated by $\#$. Moreover, every symbol in a configuration is preceded by a bit string, which represents a *counter*. By (8) every counter is non-empty, the first symbol in a configuration is preceded by a counter from 0^+, the last symbol is preceded by a counter from 1^+. Moreover, by (9), the first configuration is an initial configuration, whereas by (10), the last configuration is accepting (i.e. the state is q_f).

For the further considerations, let us fix some $x \in U_0$. Hence, we can write x as $x = x_1 \# x_2 \# \cdots x_n \#$ such that:

- For every $1 \leq i \leq n$, there exist $m_i \geq 1$, $a_{i,0}, \dots, a_{i,m_i} \in \Omega$ and counters $u_{i,0}, \dots, u_{i,m_i} \in \{0,1\}^+$ such that $x_i = \prod_{j=0}^{m_i} u_{i,j} a_{i,j}$.
- For every $1 \leq i \leq n$, $u_{i,0} \in 0^+$, $u_{i,m_i} \in 1^+$, and $\pi(x_i) \in \Gamma^* Q \Gamma^+$.
- $\pi(x_1) \in q_0 \Sigma^* \Box^*$ and $\pi(x_n) \in \Gamma^* q_f \Gamma^+$

We next want to construct, from $m \in \mathbb{N}$, a small formula expressing that x is an accepting 2^m-computation. To achieve this, we add some structure around strings from U_0. Then the formula we are seeking has to ensure two facts:

(a) The counters behave correctly, i.e. for all $1 \leq i \leq n$ and $0 \leq j \leq m_i$, we have $|u_{i,j}| = m$ and if $j < m_i$, then $\mathrm{val}(u_{i,j+1}) = \mathrm{val}(u_{i,j}) + 1$. Note that this enforces $m_i = 2^m - 1$ for all $1 \leq i \leq n$.

(b) For two successive configurations, the second one is the successor configuration of the first one, i.e., $\pi(x_i) \vdash_M \pi(x_{i+1})$ for all $1 \leq i < n$.

In order to achieve (a), we introduce the following three binary automatic relations:

$$\begin{aligned}
\sigma_0 &= \{(0v\#, v0\#) \mid v \in (\Delta \setminus \{\#\})^*\}^+ \\
\sigma_\Omega &= \Big(\{(au, ua) \mid u \in \{0,1\}^+, a \in \Omega\}^+(\#, \#)\Big)^+ \\
\delta &= \Big(\{(ua, va) \mid a \in \Omega, u, v \in \{0,1\}^+, |u| = |v|, \\
&\qquad \mathrm{val}(v) = \mathrm{val}(u) + 1 \bmod 2^{|u|}\}^+(\#, \#)\Big)^+
\end{aligned}$$

Hence, σ_0 cyclically rotates every configuration to the left for one symbol, provided the first symbol is 0, whereas σ_Ω shifts all Ω-symbols one step to the right in every configuration. The relation δ increments every counter modulo $2^{\text{length of the counter}}$. The crucial fact is the following:

Fact 1. For every $m \in \mathbb{N}$, the following two properties are equivalent (recall that $x \in U_0$):

- $\exists y_1, y_2 \in \Delta^* : \delta(x, y_2), \sigma_0^m(x, y_1), \sigma_\Omega(y_1, y_2)$.
- For all $1 \leq i \leq n$ and $0 \leq j \leq m_i$, we have $|u_{i,j}| = m$ and if $j < m_i$, then $\mathrm{val}(u_{i,j+1}) = \mathrm{val}(u_{i,j}) + 1$.

Assume now that $x \in U_0$ satisfies one (and hence both) of the two properties from Fact 1 for some m. It follows that $m_i = 2^m - 1$ for all $1 \leq i \leq n$ and

$$x = x_1\#x_2\# \cdots x_n\#, \text{ where } x_i = \prod_{j=0}^{2^m-1} \mathrm{bin}_m(j) a_{i,j} \text{ for every } 1 \leq i \leq n \ . \quad (11)$$

In order to establish (b) we need additional structure. The idea is, for every counter value $0 \leq j < 2^m$, to have a word y_j that coincides with x, but has all the occurrences of $\mathrm{bin}_m(j)$ marked. Then an automaton can check that successive occurrences of the counter $\mathrm{bin}_m(j)$ obey the transition condition of the Turing machine. There are two problems with this approach: first, in order to relate x and y_j, we would need a binary relation of degree 2^m (for arbitrary m) and, secondly, an automaton cannot mark all the occurrences of $\mathrm{bin}_m(j)$ at once (for some j). In order to solve these problems, we introduce a binary relation μ, which for every $x \in U_0$ as in (11) generates a binary tree of depth m with root x; this will be the only relation in our automatic structure that causes exponential growth. This relation will mark in x every occurrence of an arbitrary counter. For this, we need two copies $\overline{0}$ and $\underline{0}$ of 0 as well as two copies $\overline{1}$ and $\underline{1}$ of 1. For $b \in \{0,1\}$, we define the mapping $f_b : \{\underline{0}, \overline{0}, \underline{1}, \overline{1}\}^*\{0,1\}^+ \to \{\underline{0}, \overline{0}, \underline{1}, \overline{1}\}^+\{0,1\}^*$ as follows (where $u \in \{\underline{0}, \overline{0}, \underline{1}, \overline{1}\}^*$, $c \in \{0,1\}$, and $v \in \{0,1\}^*$):

$$f_b(ucv) = \begin{cases} u\underline{c}v & \text{if } b \neq c \\ u\overline{c}v & \text{if } b = c \end{cases}$$

We extend f_b to $((\{\underline{0},\overline{0},\underline{1},\overline{1}\}^*\{0,1\}^+\Omega)^+\#)^*$ as follows: For $w = w_1a_1 \cdots w_\ell a_\ell$ with $w_i \in \{\underline{0},\overline{0},\underline{1},\overline{1}\}^*\{0,1\}^+$ and $a_i \in \Omega \cup \Omega\#$ let $f_b(w) = f_b(w_1)a_1 \cdots f_b(w_\ell)a_\ell$. Since f_b can be computed with a synchronized transducer, the relation $\mu = f_0 \cup f_1$ (here f_b is viewed as a binary relation) is automatic.

Let $x \in U_0$ as in (11), let the word y be obtained from x by overlining or underlining each bit in x, and let $u \in \{0,1\}^m$ be some counter. We say *the counter u is marked in y* if every occurrence of the counter u is marked by overlining each bit, whereas all other counters contain at least one underlined bit.

Fact 2. Let $x \in U_0$ be as in (11).

- For every counter $u \in \{0,1\}^m$, there is a unique y such that $(x,y) \in \mu^m$ and u is marked in y.
- If $(x,y) \in \mu^m$, then there exists a unique counter $u \in \{0,1\}^m$ such that u is marked in y.

Now, we can achieve our final goal, namely checking whether two successive configurations in $x \in U_0$ represent a transition of the machine M. Let the counter $u \in \{0,1\}^m$ be marked in y. We describe a finite automaton A_1 that checks on the string y, whether at position $\mathrm{val}(u)$ successive configurations in x are "locally consistent". The automaton A_1 searches for the first marked counter in y. Then it stores the next three symbols a_1, a_2, a_3 from Ω (if the separator $\#$ is seen before, then only one or two symbols may be stored), walks right until it finds the next marked counter, reads the next three symbols b_1, b_2, b_3 from Ω, and checks whether $(a_1a_2a_3, b_2) \in \alpha_M$, where α_M is from (6). If this is not the case, then A_1 will reject, otherwise it will store $b_1b_2b_3$ and repeat the procedure described above. Let $U_1 = L(A_1)$. Together with Fact 1 and 2, the behavior of A_1 implies that for all $x \in U_0$ and all $m \in \mathbb{N}$, x represents an accepting 2^m-computation of M iff x satisfies the formula

$$\Phi(x) = \exists y_1, y_2\, (\delta(x,y_2) \wedge \sigma_0^m(x,y_1) \wedge \sigma_\Omega(y_1,y_2)) \wedge \forall y\, (\mu^m(x,y) \rightarrow y \in U_1)\ .$$

Let us now fix some input $w = a_1a_2 \cdots a_n \in \Sigma^*$ with $|w| = n$, and let $a_{n+1} = \Box$ and $m = 2^n$. Thus, w is accepted by M if and only if there exists an accepting 2^m-computation x such that in the first configuration of x, the tape content is of the form $w\Box^+$. It remains to add some structure that allows us to express the latter by a formula. But this is straightforward: Let $\triangleright$ be a new symbol and let

$$\Pi = \Delta \cup \{\underline{0},\overline{0},\underline{1},\overline{1},\triangleright\};$$

this is our final alphabet. Define the binary automatic relations ι_{01} and ι_a ($a \in \Omega$) as follows:

$$\begin{aligned} \iota_{01} &= \{(u \triangleright av, ua \triangleright v) \mid a \in \{0,1\}, u,v \in \Delta^*\} \cup \{(0v, 0 \triangleright v) \mid v \in \Delta^*\} \\ \iota_a &= \{(u \triangleright av, ua \triangleright v) \mid u,v \in \Delta^*\}\ . \end{aligned}$$

Then, the first configuration of x has a tape from $w\Box^+$ if and only if x satisfies the formula

$$\Psi(x) = \exists y_0, z_0, \dots, y_{n+1}, z_{n+1} \big(\iota_{01}^m(x, y_0) \ \wedge \ \iota_{q_0}(y_0, z_0) \\ \wedge \ \bigwedge_{i=1}^{n+1} \iota_{0,1}^m(z_{i-1}, y_i) \ \wedge \ \iota_{a_i}(y_i, z_i) \big).$$

Then, $\mathcal{A} = (\Pi^*, \sigma_0, \sigma_\Omega, \delta, \mu, \iota_{01}, (\iota_a)_{a \in \Omega}, U_0, U_1)$ is an automatic structure of bounded degree such that M accepts w iff the formula $\exists x \in U_0 \, (\Phi(x) \wedge \Psi(x))$ holds in $\mathcal{A}$. Lemma 4.1 allows to compute in time $O(\log(m)) = O(n)$ an equivalent formula over the signature of $\mathcal{A}$. This concludes the proof. □

The proof of the next result is in fact a simplification of the proof of Thm. 4.2, since we do not need counters. In particular, the μ-relation in the proof of Thm. 4.2, which was responsible for exponential growth, is not needed:

Theorem 4.3. *There exists a fixed automatic structure $\mathcal{A}$ of bounded degree and polynomial growth (in fact linear growth) such that* $\mathrm{Th}(\mathcal{A})$ *is* EXPSPACE*-hard.*

5 Bounded quantifier alternation depth and open problems

In this section we state some facts about first-order fragments of fixed quantifier alternation depth. These results can be deduced by reusing the construction from Section 4.

For $n \geq 0$, a Σ_n-formula is a formula in prenex normal form, where the quantifier prefix consists of n alternating blocks and the first block is a block of existential quantifiers. The Σ_n-theory of a structure $\mathcal{A}$ is the set of all Σ_n-formulas in $\mathrm{Th}(\mathcal{A})$. For a class C of automatic presentations, the *Σ_n-model checking problem* Σ_n-MC(C) *of* C denotes the set of all pairs (P, φ) where $P \in$ C, and φ belongs to the Σ_n-theory of $\mathcal{A}(P)$. The following result can be found in [3]:

Theorem 5.1 (cf. [3]). *The problem* Σ_1-MC(SA) *is in* PSPACE. *Moreover, there is a fixed automatic structure with a* PSPACE*-complete Σ_1-theory.*

From our construction in the proof of Thm. 4.3, we can slightly sharpen the lower bound in this theorem:

Theorem 5.2. *There exists a fixed automatic structure of bounded degree and polynomial growth (in fact linear growth) with a* PSPACE*-complete Σ_1-theory.*

Let us now move on to Σ_2-formulas and structures of arbitrary growth:

Theorem 5.3. Σ_2-MC(SA) *is in* EXPSPACE. *Moreover, there exists a fixed automatic structure of bounded degree with an* EXPSPACE*-complete Σ_2-theory.*

For $n \geq 3$, the precise complexity of the Σ_n-theory of an automatic structure of bounded degree remains open. From our results, it follows that the complexity is somewhere between EXPSPACE and 2EXPSPACE.

Conjecture 5.4. For $n \geq 3$, Σ_n-MC(SAb) is in EXPSPACE.

A possible attack to this conjecture would follow the line of argument in the proof of Thm. 3.5 and would therefore be based on Gaifman's theorem. To make this work, the exponential bound in Gaifman's theorem would have to be reduced which leads to the following conjecture:

Conjecture 5.5. Let $\mathcal{A}$ be a structure, $\overline{a}, \overline{b} \in \mathcal{A}^k$, and $d, n \geq 0$ such that the spheres of radius $d \cdot 2^n$ around $\overline{a}$ and $\overline{b}$ are isomorphic. Then, for every Σ_n-formula $\varphi(x_1, \ldots, x_k)$ of quantifier depth at most d, we have: $\mathcal{A} \models \varphi(\overline{a})$ iff $\mathcal{A} \models \varphi(\overline{b})$.

References

1. V. Bárány, Ł. Kaiser, and S. Rubin. Cardinality and counting quantifiers on omega-automatic structures. In *STACS'08*, pages 385–396. IFIB Schloss Dagstuhl, 2008.
2. A. Blumensath. Automatic structures. Technical report, RWTH Aachen, 1999.
3. A. Blumensath and E. Grädel. Automatic Structures. In *LICS'00*, pages 51–62. IEEE Computer Society Press, 2000.
4. K. Compton and C. Henson. A uniform method for proving lower bounds on the computational complexity of logical theories. *Ann. Pure Appl. Logic*, 48:1–79, 1990.
5. C. Elgot. Decision problems of finite automata design and related arithmetics. *Trans. Am. Math. Soc.*, 98:21–51, 1961.
6. D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson, and W. Thurston. *Word Processing In Groups*. Jones and Bartlett Publishers, Boston, 1992.
7. O. Finkel. On decidability properties of one-dimensional cellular automata. arXiv.org, 2008. `http://arxiv.org/abs/0903.4615`.
8. H. Gaifman. On local and nonlocal properties. In *Logic Colloquium '81*, pages 105–135. North-Holland, 1982.
9. B. Hodgson. On direct products of automaton decidable theories. *Theoret. Comput. Sci.*, 19:331–335, 1982.
10. B. Khoussainov and A. Nerode. Automatic presentations of structures. In *Logic and Computational Complexity*, LNCS 960, pages 367–392. Springer, 1995.
11. B. Khoussainov, S. Rubin, and F. Stephan. Definability and regularity in automatic structures. In *STACS'04*, LNCS 2996, pages 440–451. Springer, 2004.
12. D. Kuske and M. Lohrey. First-order and counting theories of ω-automatic structures. *J. Symbolic Logic*, 73:129–150, 2008.
13. D. Kuske and M. Lohrey. Automatic structures of bounded degree revisited. arXiv.org, 2008. `http://arxiv.org/abs/0810.4998`.
14. M. Lohrey. Automatic structures of bounded degree. In *LPAR'03*, LNCS 2850, pages 344–358. Springer, 2003.
15. S. Rubin. Automata presenting structures: A survey of the finite string case. *Bull. Symbolic Logic*, 14:169–209, 2008.
16. T. Rybina and A. Voronkov. Upper bounds for a theory of queues. In *ICALP'03*, Lecture Notes in Comp. Science vol. 2719, pages 714–724. Springer, 2003.
17. A. Weber. On the valuedness of finite transducers. *Acta Inform.*, 27:749–780, 1990.