Complexity of word problems for HNN-extensions

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Abstract

The computational complexity of the word problem in HNN-extension of groups is studied. HNN-extension is a fundamental construction in combinatorial group theory. It is shown that the word problem for an ascending HNN-extension of a group H is logspace reducible to the so-called compressed word problem for H. The main result of the paper states that the word problem for an HNN-extension of a hyperbolic group H with cyclic associated subgroups can be solved in polynomial time. This result can be easily extended to fundamental groups of graphs of groups with hyperbolic vertex groups and cyclic edge groups.

Keywords: Something, word problems, HNN-extensions, hyperbolic groups

1. Introduction

The study of computational problems in group theory goes back to the beginning of the 20th century. In a seminal paper from 1911, Dehn posed three decision problems [11]: The *word problem*, the *conjugacy problem*, and the *isomorphism problem*. In this paper, we mainly deal with the word problem: It is defined for a finitely generated group G. This means that there exists a finite subset $\Sigma \subseteq G$ such that every element of G can be written as a finite product of elements from Σ. This allows to represent elements of G by finite words over the alphabet Σ. For the word problem, the input consists of such a finite word $w \in \Sigma^*$ and the goal is to check whether w represents the identity element of G.

In general the word problem is undecidable. By a classical result of Boone [8] and Novikov [38], there exist finitely presented groups (finitely generated groups that can be defined by finitely many equations) with an undecidable word problem; see [45] for an excellent exposition. On the positive side, there are many classes of groups with decidable word problems. In his paper from 1912 [12], Dehn presented an algorithm that solves the word problem for fundamental groups of orientable closed 2-dimensional manifolds. This result was extended to one-relator groups (finitely generated groups that can be defined by a single equation) by Dehn's student Magnus [29]. Other important classes of groups with a decidable word problem are:

- automatic groups [16] (including important classes like braid groups [1], Coxeter groups [7], right-angled Artin groups [10], Artin groups of large

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type [23], and hyperbolic groups [18]),

- finitely generated linear groups, i.e., finitely generated groups that can be faithfully represented by matrices over a field [40] (including polycyclic groups and nilpotent groups),
- finitely generated metabelian groups (they can be embedded in direct products of linear groups [47]), and
- finitely presented residually free groups [30, 32].

With the rise of computational complexity theory in the 1960's, also the computational complexity of group theoretic problems moved into the focus of research. From the very beginning, this field attracted researchers from mathematics as well as computer science. It turned out that for many interesting classes of groups the word problem admits quite efficient algorithms. For instance, Lipton and Zalcstein [25] proved in 1977 that the word problem for a finitely generated group that is linear over a field of characteristic zero can be solved in deterministic logarithmic space. Simon [44] extended this result in 1979 to fields of prime characteristic. For automatic groups, the word problem can be solved in quadratic time [16], and for the subclass of hyperbolic groups the word problem can be solved in linear time (even real time) [21] and belongs to the complexity class LogCFL [26]. The latter is the closure of the context-free languages under logspace reductions. For one-relator groups, only a non-elementary algorithm is known for the word problem.

The complexity of the word problem is also preserved by several important group theoretic constructions, e.g., wreath products [46] and graph products [13], which generalize free products and direct products. Two other important constructions in group theory are *HNN-extensions* and *amalgamated free products*. A theorem of Seifert and van Kampen links these constructions to algebraic topology. Moreover, HNN-extensions are used in all modern proofs for the undecidability of the word problem in finitely presented groups; see e.g. [45]. For a base group H with two isomorphic subgroups A and B and an isomorphism $\varphi\colon A \to B$, the corresponding HNN-extension is the group

$$G = \langle H, t \mid t^{-1}at = \varphi(a)\ (a \in A)\rangle. \tag{1}$$

Intuitively, it is obtained by adjoining to H a new generator t (the *stable letter*) in such a way that conjugation of A by t realizes φ. The subgroups A and B are also called the *associated subgroups*. If H has a decidable word problem, A and B are finitely generated subgroups of H, and the subgroup membership problems for A and B are decidable, then also the word problem for G in (1) is decidable via *Britton reduction* [9]. In Britton reduction, one applies the following rewriting steps as long as possible:

$$t^{-1}at \to \varphi(a) \text{ for } a \in A \quad \text{ and } \quad tbt^{-1} = \varphi^{-1}(b) \text{ for } b \in B, \tag{2}$$

where a and b are represented by words over a generating set for H. For the special case where $A = B$ and φ is the identity, it is shown in [46] that the word problem for the HNN-extension G in (1) is NC^1-reducible to the following problems: (i) the word problem for H, (ii) the word problem for the free group of rank two, and (iii) the subgroup membership problem for A. On the other hand, it is not clear whether this result can be extended to arbitrary HNN-extensions (even if we allow polynomial time Turing reductions instead of NC^1-reductions). The problem with Britton reduction is that each application of a rule from (2) might increase the length of the word by a constant multiplicative factor. This might accumulate to an exponential blow-up in word length in the end. Let us consider some concrete examples:

- The Baumslag-Solitar group $\mathrm{BS}(p, q)$ (for $p, q \geq 1$) is a one-relator group with the presentation $\langle a, t \mid t^{-1}a^p t = a^q \rangle$ [5]. Britton-reduction might lead to powers a^k where $|k|$ is exponential in the length of the input word (this can only happen if one of p, q is at least 2). Still, a polynomial time algorithm for the word problem of $\mathrm{BS}(p, q)$ can be obtained by storing the exponent k in binary notation. With some more effort, Weiß [49] proved that the word problem for a Baumslag-Solitar group and, more generally, the word problem for a so-called generalized Baumslag-Solitar group (a fundamental group of a graph of groups, where all vertex groups are copies of $\mathbb{Z}$) can be even solved in logspace.
- The HNN-extension $\langle \mathrm{BS}(p, q), b \mid b^{-1}ab = t \rangle$ is known as the Baumslag group [4] (some authors call it the Baumslag-Gersten group). One can show that it is a one-relator group. A truly remarkable fact is that the Dehn function of the Baumslag group (which, roughly speaking, counts the number of relators needed to show that a trivial word of length n is the group identity) is non-elementary [39]. This is reflected in the fact that during Britton reduction, the exponents of the letters a and t may reach non-elementary size with respect to the input length. This fact let experts to the conjecture that the Baumslag group is an example of a group whose word problem cannot be solved in polynomial time. Myasnikov, Ushakov and Wong refuted this conjecture and proved that the word problem for the Baumslag group can be solved in polynomial time [35]. For this they used a tailored compressed representation of integers known as power circuits [36]. Recently, the complexity for the word problem of the Baumslag group has been further improved to NC [31].
- Finally, consider an HNN-extension $\langle F, t \mid t^{-1}at = \varphi(a)\,(a \in A)\rangle$ of a free group F with finitely generated associated subgroups A and B. The complexity of the word problem for this group is open. Note that the word problem for a free group is known to be in logspace (it is a linear group) [25] and the subgroup membership problem for finitely generated subgroups of a free group can be solved in polynomial time [2]. Still, the best known algorithm for the word problem of $\langle F, t \mid t^{-1}at = \varphi(a)\,(a \in A)\rangle$ seems to have an exponential running time.

The first two examples exploit compression of integer exponents in order to obtain polynomial time algorithms. In order to solve the open problem from the third point, one might use a suitable compressed representation of the (potentially exponentially long) words that appear during Britton reduction. Straight-line programs, i.e., context-free grammars that produce a single word, might be a good candidate for this. This idea works for the word problems of automorphism groups and certain group extensions [28, Section 4.2]. But it is not clear whether the words that arise from Britton reduction can be compressed down to polynomial size using straight-line programs. The problem arises from the fact that in (1), both A and B might be proper subgroups of H. On the other hand, if the associated subgroup A coincides with the base group H (G is then called an *ascending HNN-extension*) then one can show that the word problem for G is logspace-reducible to the so-called *compressed word problem* for H (Theorem 3.4). The latter problem has a straight-line program $\mathcal{G}$ as input, and it is asked whether the word produced by $\mathcal{G}$ evaluates to the group identity of H. The compressed word problem is known to be solvable in polynomial time for nilpotent groups [28], virtually special groups [28], and groups that are hyperbolic relative to free abelian subgroups [22]. For every linear group, one still has a randomized polynomial time algorithm for the compressed word problem [28]. Examples of groups with a PSPACE-complete word problem are Thompson's group F, the Grigorchuk group, and wreath products $G \wr \mathbb{Z}$, where G is either free of rank at least two or finite and non-solvable [3].

Our main result deals with HNN-extensions of the form (1), where the associated subgroups A and B are allowed to be proper subgroups of the base group H but are cyclic (i.e., generated by a single element) and *undistored* in H (the latter is defined in Section 5). We show that in this situation the word problem for G is polynomial time Turing-reducible to the *compressed power problem* for H (Theorem 5.1). In the compressed power problem for H, the input consists of two elements $p, q \in H$, where p is given explicitly as a word over a generating set and q is given in compressed form by a straight-line program over a generating set. The question is whether there exists an integer $z \in \mathbb{Z}$ such that $p^z = q$ in H. Moreover, in the positive case we also want to compute such a z.

Our main application of Theorem 5.1 concerns hyperbolic groups. We show that the compressed power problem for a hyperbolic group can be solved in polynomial time (Theorem 4.1). For this, we make use of the well-known fact that cyclic subgroups of hyperbolic groups are undistorted. As a consequence of Theorems 4.1 and 5.1, the word problem for an HNN-extension of a hyperbolic group with cyclic associated subgroups can be solved in polynomial time (Corollary 5.2). One should remark that HNN-extensions of hyperbolic groups with cyclic associated subgroups are in general not even automatic; a well-known example is the Baumslag-Solitar group $\mathsf{BS}(1,2) = \langle a, t \mid t^{-1}at = a^2 \rangle$ [16, Section 7.4].

Corollary 5.2 can be generalized to fundamental groups of graphs of groups (which generalize HNN-extensions and amalgamated free products) with hyperbolic vertex groups and cyclic edge groups (Corollary 6.2). For the special case where all vertex groups are free, the existence of a polynomial time algorithm for the word problem has been stated in [48, Remark 5.11] without proof.

2. Groups

For real numbers $a \leq b$ we denote with $[a, b] = \{r \in \mathbb{R} \mid a \leq r \leq b\}$ the closed interval from a to b. For $k, \ell \in \mathbb{N}$ we write $[k : \ell]$ for $\{i \in \mathbb{N} \mid k \leq i \leq \ell\}$. We use standard notations for words (over some alphabet Σ). As usual, the empty word is denoted with ε. A word u is a *factor* of a word $w \in \Sigma^*$ if there exist words $s, t \in \Sigma^*$ such that $w = sut$. If $s = \varepsilon$ (resp., $t = \varepsilon$) then u is called a *prefix* (resp., *suffix*) of w and if in addition $u \neq w$ then u is called a *proper prefix* (resp., *proper suffix*) of w. If $w = a_1 a_2 \cdots a_n$ (where $a_1, a_2, \ldots, a_n \in \Sigma$) then for all numbers $i, j \in \mathbb{N}$ with $1 \leq i \leq j$ we define the factor $w[i : j] = a_i a_{i+1} \cdots a_{\min\{j,n\}}$.

For a group G and a subset $\Sigma \subseteq G$, we denote with $\langle \Sigma \rangle$ the subgroup of G generated by Σ. It is the smallest subgroup of G containing Σ. If $G = \langle \Sigma \rangle$ then Σ is a *generating set* for G. The group G is *finitely generated (f.g.)* if it has a finite generating set. We mostly consider f.g. groups in this paper.

Assume that $G = \langle \Sigma \rangle$ and let $\Sigma^{-1} = \{a^{-1} \mid a \in \Sigma\}$. For a word $w = a_1 \cdots a_n$ with $a_i \in \Sigma \cup \Sigma^{-1}$ we define the word $w^{-1} = a_n^{-1} \cdots a_1^{-1}$. This defines an involution on the free monoid $(\Sigma \cup \Sigma^{-1})^*$. We obtain a surjective monoid homomorphism $\pi \colon (\Sigma \cup \Sigma^{-1})^* \to G$ that preserves the involution: $\pi(w^{-1}) = \pi(w)^{-1}$. We also say that the word w represents the group element $\pi(w)$. For words $u, v \in (\Sigma \cup \Sigma^{-1})^*$ we say that $u = v$ in G if $\pi(u) = \pi(v)$. For $g \in G$ one defines $|g|_\Sigma = \min\{|w| : w \in \pi^{-1}(g)\}$ as the length of a shortest word over $\Sigma \cup \Sigma^{-1}$ representing g. If Σ is clear, we also write $|g|$ for $|g|_\Sigma$. If $\Sigma = \Sigma^{-1}$ then Σ is a finite *symmetric* generating set for G.

We will describe groups by presentations. In general, if H is a group and $R \subseteq H$ is a set of so-called *relators*, then we denote with $\langle H \mid R \rangle$ the quotient group H/N_R, where N_R is the smallest normal subgroup of H with $R \subseteq N_R$. Formally, we have $N_R = \langle \{hrh^{-1} \mid h \in H, r \in R\} \rangle$. For group elements $g_i, h_i \in H$ $(i \in I)$ we also write $\langle H \mid g_i = h_i \ (i \in I) \rangle$ for the group $\langle H \mid \{g_i h_i^{-1} \mid i \in I\} \rangle$.

In most cases, one takes a free group for the group H from the previous paragraph. Fix a set Σ and let $\Sigma^{-1} = \{a^{-1} \mid a \in \Sigma\}$ be a set of formal inverses of the elements in Σ with $\Sigma \cap \Sigma^{-1} = \emptyset$. A word $w \in (\Sigma \cup \Sigma^{-1})^*$ is called *freely reduced* if it neither contains a factor aa^{-1} nor $a^{-1}a$ for $a \in \Sigma$. For every word $w \in (\Sigma \cup \Sigma^{-1})^*$ there is a unique freely reduced word $\mathrm{nf}(w)$ that is obtained from w by deleting factors aa^{-1} and $a^{-1}a$ $(a \in \Sigma)$ as long as possible in an arbitrary order (nf stands for normal form). The *free group* $F(\Sigma)$ generated by Σ consists of all freely reduced words from $(\Sigma \cup \Sigma^{-1})^*$ together with the multiplication defined by $u \cdot v = \mathrm{nf}(uv)$ for u, v freely reduced. Note that $\mathrm{nf} \colon (\Sigma \cup \Sigma^{-1})^* \to F(\Sigma)$ is a monoid morphism that preserves the involution. For a set $R \subseteq F(\Sigma)$ of relators we also write $\langle \Sigma \mid R \rangle$ for the group $\langle F(\Sigma) \mid R \rangle$. Every group G that is generated by Σ can be written as $\langle \Sigma \mid R \rangle$ for some $R \subseteq F(\Sigma)$. A group $\langle \Sigma \mid R \rangle$ with Σ and R finite is called *finitely presented*, and the pair (Σ, R) is a *presentation* for the group $\langle \Sigma \mid R \rangle$. Given two groups $G_1 = \langle \Sigma_1 \mid R_1 \rangle$ and $G_2 = \langle \Sigma_2 \mid R_2 \rangle$, where w.l.o.g. $\Sigma_1 \cap \Sigma_2 = \emptyset$, we define their *free product* $G_1 * G_2 = \langle \Sigma_1 \cup \Sigma_2 \mid R_1 \cup R_2 \rangle$.

Consider a group G with the finite symmetric generating set Σ. The *word problem for G w.r.t. Σ* is the following decision problem:

Input: a word $w \in \Sigma^*$.

Question: does $w = 1$ hold in G?

It is well known that if Σ' is another finite symmetric generating set for G, then the word problem for G w.r.t. Σ' is logspace many-one reducible to the word problem for G w.r.t. Σ. This justifies one to speak just of the word problem for the group G.

HNN-extensions. HNN-extension is an extremely important operation for constructing groups that arises in all parts of combinatorial group theory. Take a group H and a generator $t \notin H$, from which we obtain the free product $H * F(t) \cong H * \mathbb{Z}$ (we write here $F(t)$ for $F(\{t\})$). Assume that $A \leq H$

and $B \leq H$ are two isomorphic subgroups of H and let $\varphi\colon A \to B$ be an isomorphism. Then, the group

$$\langle H * F(t) \mid t^{-1}at = \varphi(a) \ (a \in A)\rangle$$

is called the *HNN-extension of A with associated subgroups A and B* (usually, the isomorphism φ is not mentioned explicitly). The above HNN-extension is usually written as

$$\langle H, t \mid t^{-1}at = \varphi(a) \ (a \in A)\rangle.$$

Britton [9] proved the following fundamental result for HNN-extensions. Let us fix a symmetric generating set Σ for H.

Theorem 2.1 (Britton's lemma [9])**.** *Let $G = \langle H, t \mid t^{-1}at = \varphi(a) \ (a \in A)\rangle$ be an HNN-extension and let $w \in (\Sigma \cup \{t, t^{-1}\})^* \setminus \Sigma^*$ be a word such that $w = 1$ in G. Then w contains a factor of the form $t^{-1}ut$ (resp., tut^{-1}), where $u \in \Sigma^*$ represents an element of A (resp., B).*

A subword of the form $t^{-1}ut$ (resp., tut^{-1}), where $u \in \Sigma^*$ represents an element of A (resp., B) is also called a *pin*.

A simple corollary of Britton's lemma is that H is a subgroup of the HNN-extension $\langle H, t \mid t^{-1}at = \varphi(a) \ (a \in A)\rangle$. Britton's lemma can be also used to solve the word problem for an HNN-extension $\langle H, t \mid t^{-1}at = \varphi(a) \ (a \in A)\rangle$. For this we need several assumptions:

- The word problem for H is decidable.
- There is an algorithm that decides whether a given word $u \in \Sigma^*$ represents an element of A (resp., B).
- Given a word $u \in \Sigma^*$ that represents an element $a \in A$ (resp., $b \in B$), one can compute a word $v \in \Sigma^*$ that represents the element $\varphi(a)$ (resp., $\varphi^{-1}(b)$). Let us denote this word v with $\varphi(u)$ (resp., $\varphi^{-1}(u)$).

Then, given a word $w \in (\Sigma \cup \{t, t^{-1}\})^*$ one replaces pins $t^{-1}ut$ (resp., tut^{-1}) by $\varphi(u)$ (resp., $\varphi^{-1}(u)$) in any order, until no more pins occur. If the final word does not belong to Σ^* then we have $w \neq 1$ in the HNN-extension. If the final word belongs to Σ^* then one uses the algorithm for the word problem of H to check whether it represents the group identity. This algorithm is known as *Britton reduction*.

If the subgroups A and B are finitely generated, say $A = \langle\{a_1, \ldots, a_n\}\rangle$ and $B = \langle\{b_1, \ldots, b_n\}\rangle$, where $\varphi(a_i) = b_i$, then the assumption from the third point concerning the computability of φ (resp., φ^{-1}) is automatically satisfied: Assume that a_i (resp., b_i) is represented by the word $u_i \in \Sigma^*$ (resp., $v_i \in \Sigma^*$). Assume that the word $u \in \Sigma^*$ represents an element $a \in A$. We then start to enumerate all words $w \in \{u_1, u_1^{-1}, \ldots, u_n, u_n^{-1}\}^*$ and check (using the word problem for H) whether $u = w$ holds in H (it is guaranteed that we will find such a word w). If this is true and $w = u_{i_1}^{\epsilon_1} \cdots u_{i_k}^{\epsilon_k}$ with $\epsilon_i \in \{-1, 1\}$ for $1 \leq i \leq k$, then $\varphi(a)$ is represented by the word $v_{i_1}^{\epsilon_1} \cdots v_{i_k}^{\epsilon_k}$. The inverse isomorphism φ^{-1} can be computed analogously.

An HNN-extension $G = \langle H, t \mid t^{-1}at = \varphi(a) \ (a \in A)\rangle$ with $\varphi\colon A \to B$ is called *ascending* if $A = H$ (it is also called the mapping torus of φ). Note that we do not require $B = H$. Ascending HNN-extensions play an important role in many group theoretical results. For instance, Bieri and Strebel [6] proved that if N is a normal subgroup of a finitely presented group G such that $G/N \cong \mathbb{Z}$ then G is an ascending HNN-extension of a finitely generated group or contains a free subgroup of rank two.

Hyperbolic groups. Let G be a f.g. group with the finite symmetric generating set Σ. The *Cayley-graph* of G (with respect to Σ) is the undirected graph $\Gamma = \Gamma(G)$ with node set G and all edges (g, ga) for $g \in G$ and $a \in \Sigma$. We view Γ as a geodesic metric space, where every edge (g, ga) is identified with a unit-length interval. It is convenient to label the directed edge from g to ga with the generator a. The distance between two points p, q is denoted with $d_\Gamma(p, q)$. Note that $|g|_\Sigma = d_\Gamma(1, g)$ for $g \in G$. For $r \geq 0$, let $\mathcal{B}_r(1) = \{g \in G \mid d_\Gamma(1, g) \leq r\}$ be the ball of radius r around the group identity.

Paths can be defined in a very general way for metric spaces, but we only need paths that are induced by words over Σ. Given a word $w \in \Sigma^*$ of length n, one obtains a unique path $P[w] \colon [0, n] \to \Gamma$, which is a continuous mapping from the real interval $[0, n]$ to Γ. It maps the subinterval $[i, i+1] \subseteq [0, n]$ with $i \in \mathbb{N}$ isometrically onto the edge (g_i, g_{i+1}) of Γ, where g_i (resp., g_{i+1}) is the group element represented by the word $w[1 : i]$ (resp., $w[1 : i+1]$). The path $P[w]$ starts in $1 = g_0$ and ends in g_n (the group element represented by w). We also say that $P[w]$ is the unique path that starts in 1 and is labelled with the word w. More generally, for $g \in G$ we denote with $g \cdot P[w]$ the path that starts in g and is labelled with w. When writing $u \cdot P[w]$ for a word $u \in \Sigma^*$, we mean the path $g \cdot P[w]$, where g is the group element represented by u.

Let $\lambda, \zeta > 0$, $\epsilon \geq 0$ be real constants. A path $P \colon [0, n] \to \Gamma$ of the above form is *geodesic* if $d_\Gamma(P(0), P(n)) = n$; it is a (λ, ϵ)-*quasigeodesic* if for all points $p = P(a)$ and $q = P(b)$ with $a, b \in [0, n]$ we have $|a - b| \leq \lambda \cdot d_\Gamma(p, q) + \epsilon$; and it is ζ-*local* (λ, ϵ)-*quasigeodesic* if for all points $p = P(a)$ and $q = P(b)$ with $a, b \in [0, n]$ and $|a - b| \leq \zeta$ we have $|a - b| \leq \lambda \cdot d_\Gamma(p, q) + \epsilon$.

A word $w \in \Sigma^*$ is geodesic if the path $P[w]$ is geodesic, which means that there is no shorter word representing the same group element from G. Similarly, we define the notion of (ζ-local) (λ, ϵ)-quasigeodesic words. A word $w \in \Sigma^*$ is *shortlex reduced* if it is the length-lexicographically smallest word that represents the same group element as w. For this, we have to fix an arbitrary linear order on Σ. Note that if $u = xy$ is shortlex reduced then x and y are shortlex reduced too. For a word $u \in \Sigma^*$ we denote with $\mathrm{shlex}(u)$ the unique shortlex reduced word that represents the same group element as u (the underlying group G will be always clear from the context).

A *geodesic triangle* in G consists of three points $p, q, r \in G$ and geodesic paths $P_1 = P_{p,q}$, $P_2 = P_{p,r}$, $P_3 = P_{q,r}$ (the three sides of the triangle), where $P_{x,y}$ is a geodesic path from x to y. We call a geodesic triangle δ-*slim* for $\delta \geq 0$, if for all $i \in \{1, 2, 3\}$, every point in $\mathrm{im}(P_i)$ (the image of the path $P_i \colon [0, n] \to \Gamma$) has distance at most δ from a point in $\mathrm{im}(P_j) \cup \mathrm{im}(P_k)$, where $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$. The group G is called δ-*hyperbolic*, if every geodesic triangle is δ-slim. Finally, G is hyperbolic, if it is δ-hyperbolic for some $\delta \geq 0$. Finitely generated free groups are for instance 0-hyperbolic. The property of being hyperbolic is independent of the chosen generating set Σ. Hyperbolic groups were introduced by Gromov [18].

Fix a δ-hyperbolic group G with the finite symmetric generating set Σ for the rest of the section, and let Γ be the corresponding geodesic metric space. Let

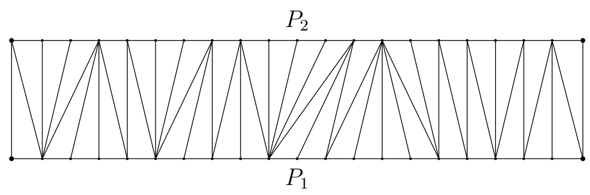

Figure 1: Paths that asynchronously κ-fellow travel

us write $|g|$ for $|g|_\Sigma$. The word problem for G can be decided in real time [21]. Moreover, for a given word $u \in \Sigma^*$, $\mathrm{shlex}(u)$ can be computed in polynomial time [16, Theorem 2.3.10]. We also need the following lemma:

Lemma 2.2 (c.f. [17, Proposition 8.21])**.** *Let $g \in G$ be of infinite order and let $n \geq 0$. Let u be a geodesic word representing g. Then the word u^n is (λ, ϵ)-quasigeodesic, where $\lambda = N|g|$, $\epsilon = 2N^2|g|^2 + 2N|g|$ and $N = |\mathcal{B}_{2\delta}(1)|$.*

Consider two paths $P_1\colon [0, n_1] \to \Gamma$, $P_2\colon [0, n_2] \to \Gamma$ and let $\kappa \in \mathbb{R}$, $\kappa \geq 0$. The paths P_1 and P_2 *asynchronously κ-fellow travel* if there exist two continuous non-decreasing mappings $\varphi_1\colon [0,1] \to [0, n_1]$ and $\varphi_2\colon [0,1] \to [0, n_2]$ such that $\varphi_1(0) = \varphi_2(0) = 0$, $\varphi_1(1) = n_1$, $\varphi_2(1) = n_2$ and for all $0 \leq t \leq 1$, $d_\Gamma(P_1(\varphi_1(t)), P_2(\varphi_2(t))) \leq \kappa$. Intuitively, this means that one can travel along the paths P_1 and P_2 asynchronously with variable speeds such that at any time instant the current points have distance at most κ. If P_1 and P_2 asynchronously κ-fellow travel, then by slightly increasing κ one obtains a subset $E \subseteq [0 : n_1] \times [0 : n_2]$ with (i) $(0,0), (n_1, n_2) \in E$, $d_\Gamma(P_1(i), P_2(j)) \leq \kappa$ for all $(i,j) \in E$ and (iii) if $(i,j) \in E \setminus \{(n_1,n_2)\}$ then $(i+1, j) \in E$ or $(i, j+1) \in E$. We write $P_1 \approx_\kappa P_2$ in this case. Intuitively, this means that a ladder graph as shown in Figure 1 exists, where the edges connecting points from the paths P_1 and P_2 represent paths of length at most κ that connect elements from G.

Lemma 2.3 (c.f.[34, Lemma 1])**.** *Let P_1 and P_2 be (λ, ϵ)-quasigeodesic paths in Γ and assume that P_i starts in g_i, ends in h_i, and $d_\Gamma(g_1, g_2), d_\Gamma(h_1, h_2) \leq h$. Then there is a constant $\kappa = \kappa(\delta, \lambda, \epsilon, h) \geq h$ such that $P_1 \approx_\kappa P_2$.*

For the following lemmas we fix two further constants:

$$L = 34\delta + 2 \quad \text{and} \quad K = |\mathcal{B}_{4\delta}(1)|^2. \tag{3}$$

Lemma 2.4 (c.f. [15, Lemma 3.1])**.** *Let $u = u_1 u_2$ be shortlex reduced, where $|u_1| \leq |u_2| \leq |u_1| + 1$. Let $\tilde{u} = \mathrm{shlex}(u_2 u_1)$. If $|\tilde{u}| \geq 2L+1$ then for every $n \geq 0$, the word $\tilde{u}^n$ is L-local $(1, 2\delta)$-quasigeodesic.*

The following lemma is not stated explicitly in [15] but is shown in [15, Section 3.2] (where the main argument is attributed to Delzant).

Lemma 2.5 (c.f. [15, Section 3.2])**.** *Let u be geodesic such that $|u| \geq 2L + 1$ and for every $n \geq 0$, the word u^n is L-local $(1, 2\delta)$-quasigeodesic. Then, in time $\mathcal{O}(|u|)$ one can compute $c \in \mathcal{B}_{4\delta}(1)$ and an integer $1 \leq m \leq K$ such that $(\mathrm{shlex}(c^{-1} u^m c))^n$ is geodesic for all $n \geq 0$.*

3. Compressed words and the compressed word problem

Straight-line programs offer succinct representations of long words that contain many repeated substrings. We review here the basics, referring to [28] for a more in-depth introduction.

Fix a finite alphabet Σ. A *straight-line program* $\mathcal{G}$ (SLP for short) is a context-free grammar that generates exactly one word $\mathrm{val}(\mathcal{G}) \in \Sigma^*$. More formally, an SLP over Σ is a triple $\mathcal{G} = (V, S, \rho)$ where

- V is a finite set of *variables*, disjoint from Σ,
- $S \in V$ is the *start variable*, and
- $\rho\colon V \to (V \cup \Sigma)^*$ is the *right-hand side mapping*, which is acyclic in the sense that the binary relation $\{(A, B) \in V \times V \mid B \text{ appears in } \rho(A)\}$ is acyclic.

We define the *size* $|\mathcal{G}|$ of $\mathcal{G}$ as $\sum_{A \in V} |\rho(A)|$. The *evaluation* function

$$\mathrm{val}_{\mathcal{G}}\colon (V \cup \Sigma)^* \to \Sigma^*$$

is the unique homomorphism between free monoids such that

- $\mathrm{val}_{\mathcal{G}}(a) = a$ for all $a \in \Sigma$ and
- $\mathrm{val}_{\mathcal{G}}(A) = \mathrm{val}_{\mathcal{G}}(\rho(A))$ for all $A \in V$.

Note that $\mathrm{val}_{\mathcal{G}}$ is well-defined since ρ is acyclic. We also write val for $\mathrm{val}_{\mathcal{G}}$ if $\mathcal{G}$ is clear from the context. We finally take $\mathrm{val}(\mathcal{G}) = \mathrm{val}(S)$. We call $\mathrm{val}(\mathcal{G})$ the word defined by the SLP $\mathcal{G}$.

Example 3.1. *Let $\Sigma = \{a, b\}$ and fix $n \geq 0$. We define*

$$\mathcal{G}_n = (\{A_0, \ldots, A_n\}, A_n, \rho),$$

where $\rho(A_0) = ab$ and $\rho(A_{i+1}) = A_i A_i$ for $0 \leq i \leq n-1$. It is an SLP of size $2(n+1)$. We have $\mathrm{val}(A_0) = ab$ and more generally $\mathrm{val}(A_i) = (ab)^{2^i}$. Thus $\mathrm{val}(\mathcal{G}_n) = \mathrm{val}(A_n) = (ab)^{2^n}$.

Example 3.1 shows that an SLP can define a word whose length is exponential in the size of the SLP. In this sense, an SLP can be seen as a compressed representation of a word. Indeed, SLPs have been intensively studied in the context of data compression; see [27] for more details.

The SLP $\mathcal{G} = (V, S, \rho)$ is *trivial* if S is the only variable and $\rho(S) = \varepsilon = \mathrm{val}(\mathcal{G})$. An SLP is in *Chomsky normal form* if it is either trivial or all right-hand sides $\rho(A)$ are of the form $a \in \Sigma$ or BC with $B, C \in V$. There is a linear-time algorithm that transforms a given SLP $\mathcal{G}$ into an SLP $\mathcal{G}'$ in Chomsky normal such that $\mathrm{val}(\mathcal{G}) = \mathrm{val}(\mathcal{G}')$; see [28, Proposition 3.8].

The following theorem is the technical main result from [24]:

Theorem 3.2 (c.f. [24])**.** *Let G be a hyperbolic group with the finite symmetric generating set Σ. Given an SLP $\mathcal{G}$ over Σ one can compute in polynomial time an SLP $\mathcal{H}$ over Σ such that $\mathrm{val}(\mathcal{H}) = \mathrm{shlex}(\mathrm{val}(\mathcal{G}))$.*

If G is a f.g. group with the finite and symmetric generating set Σ, then we define the *compressed word problem* of G as the following decision problem:

Input: an SLP $\mathcal{G}$ over Σ.

Question: does $\mathrm{val}(\mathcal{G})$ represent the group identity of G?

An immediate consequence of Theorem 3.2 is the following result:

Theorem 3.3 (c.f. [24])**.** *The compressed word problem for a hyperbolic group can be solved in polynomial time.*

The compressed word problem turns out to be useful for the solution of the word problem for an ascending HNN-extension. The following result has been stated by Schleimer [42, Remark 4.2] for the case that H is a f.g. free group, but the proof can be generalized to an arbitrary f.g. group H.

Theorem 3.4. *Let H be a f.g. group. The word problem for an ascending HNN-extension $G = \langle H, t \mid t^{-1}at = \varphi(a)\ (a \in H)\rangle$ is logspace-reducible to the compressed word problem for H.*

Proof. The proof is similar to corresponding results for automorphism groups and semi-direct products [28, Section 4.2]. Let us fix a finite and (w.l.o.g.) symmetric generating set Σ for H and a homomorphism $\tilde{\varphi}\colon \Sigma^* \to \Sigma^*$ such that for every $a \in \Sigma$, the word $\tilde{\varphi}(a)$ represents the group element $\varphi(a) \in G$.

Consider an input word $w \in (\Sigma \cup \{t, t^{-1}\})^*$ and write

$$w = w_0 t^{\epsilon_1} w_1 t^{\epsilon_2} w_2 \cdots t^{\epsilon_n} w_n,$$

where $w_i \in \Sigma^*$ for $0 \le i \le n$ and $\epsilon_i \in \{-1, 1\}$ for $1 \le i \le n$. Let $s_k = \sum_{i=1}^{k} \epsilon_i$ for $0 \le k \le n$ (in particular, $s_0 = 0$). Clearly, $w = 1$ in G if and only if $t^{-n} w t^n = 1$ in G if and only if

$$t^{-n}\left(\prod_{i=0}^{n} t^{s_i} w_i t^{-s_i}\right) t^{s_n+n} = \left(\prod_{i=0}^{n} t^{s_i-n} w_i t^{n-s_i}\right) t^{s_n} = \left(\prod_{i=0}^{n} \tilde{\varphi}^{n-s_i}(w_i)\right) t^{s_n} = 1 \tag{4}$$

in G. By Britton's lemma, (4) is equivalent to $s_n = 0$ (this can be checked in logspace) and $\prod_{i=0}^{n} \varphi^{n-s_i}(w_i) = 1$ in H. The latter is an instance of the compressed word problem. We can easily (in logspace) compute an SLP for the word $\prod_{i=0}^{n} \tilde{\varphi}^{n-s_i}(w_i)$, see e.g. [28, Lemma 3.12]. ☐

We will also need a generalization of straight-line programs, known as composition systems [19, Definition 8.1.2] (in [28] they are called *cut straight-line programs*). A *composition system* over Σ is a tuple $\mathcal{G} = (V, S, \rho)$, with V and S as for an SLP, and where we also allow, as right-hand sides for ρ, expressions of the form $B[i : j]$, with $B \in V$ and $i, j \in \mathbb{N}$, $1 \le i \le j$. The numbers i and j are stored in binary encoding. We again require ρ to be acyclic. When $\rho(A) = B[i : j]$ we define $\mathrm{val}(A) = \mathrm{val}(B)[i : j]$. We define the *size* $|\mathcal{G}|$ of the composition system $\mathcal{G}$ as the total number of occurrences of symbols from $V \cup \Sigma$ in all right-hand sides. Hence, a right-hand $B[i : j]$ contributes 1 to the size, and we ignore the numbers i, j. Adding the bit lengths of the numbers i and j to the size $|\mathcal{G}|$ would only lead to a polynomial blow-up for $|\mathcal{G}|$. To see this, first normalize the composition system so that all right-hand sides have the form

a, BC or $B[i:j]$ with $a \in \Sigma$ and $B, C \in V$. Analogously to the Chomsky normal form for SLPs, this can be achieved in linear time. If n is the number of variables of the resulting composition system, then every variable produces a string of length at most 2^n. Hence, we can assume that all numbers i, j that appear in a right-hand side $B[i:j]$ are of bit length $\mathcal{O}(n)$.

We can now state an important result of Hagenah; see [19, Algorithmus 8.1.4] as well as [28, Theorem 3.14].

Theorem 3.5 (c.f. [19])**.** *There is a polynomial-time algorithm that, given a composition system $\mathcal{G}$, computes an SLP $\mathcal{G}'$ such that* $\mathrm{val}(\mathcal{G}) = \mathrm{val}(\mathcal{G}')$.

It will be convenient to allow in composition systems also more complex right-hand sides. For instance $(ABC)[i:j]D$ would first concatenate the strings produced from A, B, and C. From the resulting string the substring from position i to position j is cut out and this substring is concatenated with the string produced by D.

4. The compressed power problem

In Section 5 we will study the word problem in HNN-extensions with cyclic associated subgroups. For this, the following computational problem turns out to be important. Let G be a f.g. group with the finite symmetric generating set Σ. We define the *compressed power problem* for G as the following problem:

Input: a word $w \in \Sigma^*$ and an SLP $\mathcal{G}$ over Σ.

Output: the binary encoding of an integer $z \in \mathbb{Z}$ such that $w^z = \mathrm{val}(\mathcal{G})$ in G if such an integer exists, and *no* otherwise.

The main result of this section is:

Theorem 4.1. *For every hyperbolic group G, the compressed power problem can be solved in polynomial time.*

Proof. Assume that G is δ-hyperbolic. Fix the word $w \in \Sigma^*$ and the SLP $\mathcal{G} = (V, \rho, S)$ over Σ. W.l.o.g. assume that $\mathcal{G}$ is in Chomsky normal form. We have to check whether the equation

$$w^z = \mathrm{val}(\mathcal{G}) \tag{5}$$

has a solution in G, and compute in the positive case a solution $z \in \mathbb{Z}$. Let g be the group element represented by w.

In a hyperbolic group G, the order of torsion elements is bounded by a fixed constant that only depends on G, see also the proof of [37, Theorem 6.7]. Since the word problem for G can be decided in linear time, we can check also in linear time whether g has finite order in G. If g has finite order, say d, then it remains to check for all $0 \le i \le d-1$ whether $w^i = \mathrm{val}(\mathcal{G})$ in G, which can be done in polynomial time by Theorem 3.3. This solves the case where g has finite order in G.

Now assume that g has infinite order in G. Then (5) has at most one solution. By considering also the equation $(w^{-1})^z = \mathrm{val}(\mathcal{G})$, it suffices to search for a solution $z \in \mathbb{N}$. We can also assume that w is shortlex-reduced. Using techniques from [15] one can further ensure that for every $n \in \mathbb{N}$, w^n is (λ, ϵ)-quasigeodesic for fixed constants λ and ϵ that only depend on the group G:

Reduction to the case with w^n (λ,ϵ)-quasigeodesic for all n. Let us fix the two constants L and K from (3) and define further constants:

$$N = |\mathcal{B}_{2\delta}(1)|, \quad \lambda = N(2L+1), \quad \text{and} \quad \epsilon = 2N^2(2L+1)^2 + 2N(2L+1). \tag{6}$$

We factorize w uniquely as $w = uv$ where $|u| \le |v| \le |u|+1$, and let $\tilde{w} = \mathrm{shlex}(vu)$. Note that $|\tilde{w}| \le |w|$. Let $\tilde{g}$ be the group element represented by $\tilde{w}$. Since $\tilde{g}$ is conjugated to g, also $\tilde{g}$ has infinite order. By Lemma 2.2, for every $n \ge 0$, the word $\tilde{w}^n$ is (λ',ϵ')-quasigeodesic for $\lambda' = N|\tilde{w}|$, $\epsilon' = 2N^2|\tilde{w}|^2+2N|\tilde{w}|$. If $|\tilde{w}| < 2L+1$ then $\tilde{w}^n$ is (λ,ϵ)-quasigeodesic for the constants λ and ϵ from (6). We then replace the equation $w^z = \mathrm{val}(\mathcal{G})$ in (5) by the equivalent equation $u\tilde{w}^z u^{-1} = \mathrm{val}(\mathcal{G})$ (or $\tilde{w}^z = u^{-1}\mathrm{val}(\mathcal{G})u$). To see the equivalence of these two equations, note that for every $n \ge 0$, $u\tilde{w}^n u^{-1} = u(vu)^n u^{-1} = (uv)^n = w^n$ in G.

Now assume that $|\tilde{w}| \ge 2L+1$. By Lemma 2.4, $\tilde{w}^n$ is L-local $(1,2\delta)$-quasigeodesic for every $n \ge 0$. By Lemma 2.5, one can compute in time $\mathcal{O}(|w|)$ an element $c \in \mathcal{B}_{4\delta}(1)$ and an integer $1 \le m \le K$ such that $(\mathrm{shlex}(c^{-1}\tilde{w}^m c))^n$ is geodesic (and hence $(1,0)$-quasigeodesic) for all $n \ge 0$. We then produce for every number $0 \le d \le m-1$ a new equation $u\tilde{w}^d c(\mathrm{shlex}(c^{-1}\tilde{w}^m c))^z c^{-1}u^{-1} = \mathrm{val}(\mathcal{G})$, or, equivalently, $(\mathrm{shlex}(c^{-1}\tilde{w}^m c))^z = c^{-1}\tilde{w}^{-d}u^{-1}\mathrm{val}(\mathcal{G})uc$. Let us denote this equation with $\mathcal{E}_d$. Then the following holds:

- if $w^n = \mathrm{val}(\mathcal{G})$ in G for $n \in \mathbb{N}$ then $\lfloor n/m \rfloor$ is a solution of $\mathcal{E}_d$, where $d = n \bmod m$, and
- if n is a solution of $\mathcal{E}_d$ for some $0 \le d \le m-1$, then $w^{n\cdot m+d} = \mathrm{val}(\mathcal{G})$ in G.

Hence, it suffices to check for each of the constantly many equations $\mathcal{E}_d$ ($0 \le d \le m-1$) whether it has a solution and to compute the solution if it exists.

The above consideration shows that we can restrict to the case of an equation $w^z = \mathrm{val}(\mathcal{G})$, where w represents a group element of infinite order and for every $n \in \mathbb{N}$, w^n is (λ,ϵ)-quasigeodesic for fixed constants λ and ϵ.

Finally, by Theorem 3.2 we can also assume that the word $\mathrm{val}(\mathcal{G})$ (and hence every word $\mathrm{val}(X)$ for X a variable of $\mathcal{G}$) is shortlex-reduced. Hence, if $w^z = \mathrm{val}(\mathcal{G})$ for some $z \in \mathbb{N}$, then by Lemma 2.3 we have $P[w^z] \approx_\kappa P[\mathrm{val}(\mathcal{G})]$ for a fixed constant κ that only depends on G. We proceed in two steps.

Step 1. We compute in polynomial time for all variables $X \in V$ of the SLP $\mathcal{G}$, all group elements $a, b \in \mathcal{B}_\kappa(1)$ (there are only constantly many), and all factors w' of w a bit $\beta[X,a,b,w'] \in \{0,1\}$ which is defined by:

$$\beta[X,a,b,w'] = \begin{cases} 1 & \text{if } \mathrm{val}(X) = aw'b \text{ in } G \text{ and } P[\mathrm{val}(X)] \approx_\kappa a \cdot P[w'] \\ 0 & \text{otherwise} \end{cases}$$

We compute these bits $\beta[X,a,b,w']$ in a bottom-up process where we begin with variables X such that $\rho(X)$ is a terminal symbol and end with the start variable S. So, let us start with a variable X such that $\rho(X) = c \in \Sigma$ and let a,b,w' as above. Then we have to check whether $c = aw'b$ in G and $P[c] \approx_\kappa a \cdot P[w']$. The former can be checked in linear time (it is an instance of the word problem) and the latter can be done in polynomial time as well: we have to check whether the path $a \cdot P[w']$ splits into two parts, where all vertices in the first (resp.,

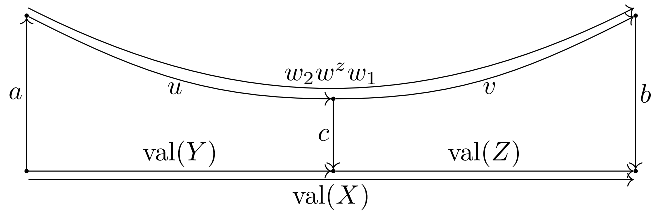

Figure 2: Situation in the proof of Lemma 4.1.

second) part belong to $\mathcal{B}_\kappa(1)$ (resp., $\mathcal{B}_\kappa(c)$). This can be reduced to $\mathcal{O}(|w'|)$ many instances of the word problem.

Let us now consider a variable X with $\rho(X) = YZ$ such that all bits $\beta[Y, a, b, w']$ and $\beta[Z, a, b, w']$ have been computed. Let us fix $a, b \in \mathcal{B}_\kappa(1)$ and a factor w' of w. We have $\beta[X, a, b, w'] = 1$ if and only if there exists a factorization $w' = w'_1 w'_2$ and $c \in \mathcal{B}_\kappa(1)$ such that $\beta[Y, a, c, w'_1] = 1$ and $\beta[Z, c^{-1}, b, w'_2] = 1$. This allows us to compute $\beta[X, a, b, w']$ in polynomial time.

Step 2. We compute in polynomial time for all variables $X \in V$, all group elements $a, b \in \mathcal{B}_\kappa(1)$, all proper suffixes w_2 of w, and all proper prefixes w_1 of w the unique number $z = z[X, a, b, w_2, w_1] \in \mathbb{N}$ (if it exists) such that

- $\mathrm{val}(X) = aw_2 w^z w_1 b$ in G and
- $P[\mathrm{val}(X)] \approx_\kappa a \cdot P[w_2 w^z w_1]$.

If such an integer z does not exist we set $z[X, a, b, w_2, w_1] = \infty$. Note that the integers $z[X, a, b, w_2, w_1]$ are unique since w represents a group element of infinite order. We represent $z[X, a, b, w_2, w_1]$ in binary encoding. As in step 1, the computation of the numbers $z[X, a, b, w_2, w_1]$ begins with variables X such that $\rho(X)$ is a terminal symbol and ends with the start variable S. The bits $\beta[X, a, b, w']$ from step 1 are needed in the computation.

Let us start with a variable X such that $\rho(X) = c \in \Sigma$ and let a, b, w_2, w_1 as above. We have to consider the equation $c = aw_2 w^z w_1 b$, or, equivalently, $w^z = u$ where $u = \mathrm{shlex}(w_2^{-1} a^{-1} c b^{-1} w_1^{-1})$. We can compute the word u in polynomial time. Since w^n is (λ, ϵ)-quasigeodesic for all $n \in \mathbb{N}$, every $n \in \mathbb{N}$ with $w^n = u$ in G has to satisfy $n \cdot |w| \leq \lambda \cdot |u| + \epsilon$, i.e., $n \leq |w|^{-1}(\lambda \cdot |u| + \epsilon)$. Hence, we can check for all $0 \leq n \leq |w|^{-1}(\lambda \cdot |u| + \epsilon)$ whether $w^n = u$ in G. If we do not find a solution, we set $z[X, a, b, w_1, w_2] = \infty$. If we find a (unique) solution n, we can check in polynomial time whether $P[\mathrm{val}(X)] = P[c] \approx_\kappa a \cdot P[w_2 w^n w_1]$ as above for $P[\mathrm{val}(X)] \approx_\kappa a \cdot P[w']$ in step 1.

Now, let X be a variable with $\rho(X) = YZ$ such that all values $z[Y, a, b, w_2, w_1]$ and $z[Z, a, b, w_2, w_1]$ have been computed. Let us fix $a, b \in \mathcal{B}_\kappa(1)$, a proper suffix w_2 of w and a proper prefix w_1 of w. Note that if $\mathrm{val}(X) = aw_2 w^z w_1 b$ in G and $P[\mathrm{val}(X)] \approx_\kappa a \cdot P[w_2 w^z w_1]$ for some $z \in \mathbb{N}$, then there must exist $c \in \mathcal{B}_\kappa(1)$ and a factorization $w_2 w^z w_1 = uv$ such that

- $\mathrm{val}(Y) = auc$ and $\mathrm{val}(Z) = c^{-1} v b$ in G,
- $P[\mathrm{val}(Y)] \approx_\kappa a \cdot P[u]$, and
- $P[\mathrm{val}(Z)] \approx_\kappa c^{-1} \cdot P[v]$; see Figure 2.

For the factorization $w_2 w^z w_1 = uv$, one of the following cases has to hold:

- There is a factorization $w_2 = uw_2'$ such that $v = w_2' w^z w_1$. We then have $\beta[Y, a, c, u] = 1$ and $z = z[Z, c^{-1}, b, w_2', w_1]$. Vice versa, if $\beta[Y, a, c, u] = 1$, $z[Z, c^{-1}, b, w_2', w_1] < \infty$ and $w_2 = uw_2'$ then
$$z[X, a, b, w_2, w_1] = z[Z, c^{-1}, b, w_2', w_1].$$
- There is a factorization $w_1 = w_1' v$ such that $u = w_2 w^z w_1'$. We then have $z = z[Y, a, c, w_2, w_1']$ and $\beta[Z, c^{-1}, b, v] = 1$. Vice versa, if $z[Y, a, c, w_2, w_1'] < \infty$, $\beta[Z, c^{-1}, b, v] = 1$ and $w_1 = w_1' v$ then
$$z[X, a, b, w_2, w_1] = z[Y, a, c, w_2, w_1'].$$
- There are $z_1, z_2 \in \mathbb{N}$ such that $u = w_2 w^{z_1}$, $v = w^{z_2} w_1$, and $z = z_1 + z_2$. We then have $z = z[Y, a, c, w_2, \varepsilon] + z[Z, c^{-1}, b, \varepsilon, w_1]$. Vice versa, if $z[Y, a, c, w_2, \varepsilon] < \infty$ and $z[Z, c^{-1}, b, \varepsilon, w_1] < \infty$ then
$$z[X, a, b, w_2, w_1] = z[Y, a, c, w_2, \varepsilon] + z[Z, c^{-1}, b, \varepsilon, w_1].$$
- There are $z_1, z_2 \in \mathbb{N}$ and a factorization $w = w_1' w_2'$ such that $w_1' \neq \varepsilon \neq w_2'$, $u = w_2 w^{z_1} w_1'$, $v = w_2' w^{z_2} w_1$, and $z = z_1 + z_2 + 1$. We then have $z = z[Y, a, c, w_2, w_1'] + z[Z, c^{-1}, b, w_2', w_1] + 1$. Vice versa, if $z[Y, a, c, w_2, w_1'] < \infty$, $z[Z, c^{-1}, b, w_2', w_1] < \infty$ and $w = w_1' w_2'$ then
$$z[X, a, b, w_2, w_1] = z[Y, a, c, w_2, w_1'] + z[Z, c^{-1}, b, w_2', w_1] + 1.$$

From these observations it is straightforward to compute in polynomial time all values $z[X, a, b, w_2, w_1]$ from the values $z[Y, a, c, w_2, w_1']$, $z[Z, c^{-1}, b, w_2', w_1]$, $\beta[Y, a, c, u]$, $\beta[Z, c^{-1}, b, v]$, where $c \in \mathcal{B}_\kappa(1)$, w_1' is a proper prefix of w, w_2' is a proper suffix of w, and u and v are factors of w.

Finally, if $z[S, 1, 1, \varepsilon, \varepsilon] = \infty$ then equation (5) has no solution, otherwise $z[S, 1, 1, \varepsilon, \varepsilon]$ is the unique solution of equation (5). This completes the proof of the theorem. □

5. HNN-extensions with cyclic associated subgroups

Let H be a f.g. group and fix a generating set Σ for H. We say that a cyclic subgroup $\langle g \rangle \leq H$ is *undistorted* in H if there exists a constant δ such that for every $h \in \langle g \rangle$ there exists $z \in \mathbb{Z}$ with $h = g^z$ and $|z| \leq \delta \cdot |h|_\Sigma$. This definition does not depend on the choice of Σ.[1] Clearly, every finite cyclic subgroup $\langle g \rangle$ of H is undistorted.

Note that if $g, h \in H$ are elements of the same order then the group $\langle H, t \mid t^{-1} g t = h \rangle$ is the HNN-extension $\langle H, t \mid t^{-1} a t = \varphi(a) \ (a \in \langle g \rangle) \rangle$, where $\varphi \colon \langle g \rangle \to \langle h \rangle$ is the isomorphism with $\varphi(g^z) = h^z$ for all $z \in \mathbb{Z}$. In the following theorem we consider a slight extension of the word problem for such an HNN-extension $G = \langle H, t \mid t^{-1} g t = h \rangle$ which we call the *semi-compressed word problem* for G. In

[1] The concept of undistorted subgroups is defined for arbitrary finitely generated subgroups but we will need it only for the cyclic case.

this problem the input is a sequence $\mathcal{G}_0 t^{\epsilon_1} \mathcal{G}_1 t^{\epsilon_2} \mathcal{G}_2 \cdots t^{\epsilon_n} \mathcal{G}_n$ where every $\mathcal{G}_i$ ($0 \leq i \leq n$) is an SLP (or a composition system) over the alphabet Σ and $\epsilon_i \in \{-1, 1\}$ for $1 \leq i \leq n$. The question is whether $\mathrm{val}(\mathcal{G}_0) t^{\epsilon_1} \mathrm{val}(\mathcal{G}_1) t^{\epsilon_2} \mathrm{val}(\mathcal{G}_2) \cdots t^{\epsilon_n} \mathrm{val}(\mathcal{G}_n) = 1$ in G.

Theorem 5.1. *Let H be a fixed f.g. group and let $g, h \in H$ be elements with the same order in H (so that the cyclic subgroups $\langle g \rangle$ and $\langle h \rangle$ are isomorphic) such that $\langle g \rangle$ and $\langle h \rangle$ are undistorted. Then the semi-compressed word problem for the HNN-extension $\langle H, t \mid t^{-1} g t = h \rangle$ is polynomial-time Turing-reducible to the compressed power problem for H.*

Proof. The case where $\langle g \rangle$ and $\langle h \rangle$ are both finite is easy. In this case, by the main result of [20], even the compressed word problem for $\langle H, t \mid t^{-1} g t = h \rangle$ is polynomial time Turing-reducible to the compressed word problem for H, which is a special case of the compressed power problem.

Let us now assume that $\langle g \rangle$ and $\langle h \rangle$ are infinite. Fix a symmetric finite generating set Σ for H. Let $W = \mathcal{G}_0 t^{\epsilon_1} \mathcal{G}_1 t^{\epsilon_2} \mathcal{G}_2 \cdots t^{\epsilon_n} \mathcal{G}_n$ be an input for the semi-compressed word problem for $\langle H, t \mid t^{-1} g t = h \rangle$, where $\mathcal{G}_i$ is a composition system over Σ for $0 \leq i \leq n$ and $\epsilon_i \in \{-1, 1\}$ for $1 \leq i \leq n$. Basically, we do Britton reduction in any order on the word $\mathrm{val}(\mathcal{G}_0) t^{\epsilon_1} \mathrm{val}(\mathcal{G}_1) t^{\epsilon_2} \mathrm{val}(\mathcal{G}_2) \cdots t^{\epsilon_n} \mathrm{val}(\mathcal{G}_n)$. The number of Britton reduction steps is bounded by $n/2$. After the i-th step we have a sequence

$$U = \mathcal{H}_0 t^{\zeta_1} \mathcal{H}_1 t^{\zeta_2} \mathcal{H}_2 \cdots t^{\zeta_m} \mathcal{H}_m,$$

where $m \leq n$, $\mathcal{H}_i = (V_i, S_i, \rho_i)$ is a composition system over Σ, and $\zeta_i \in \{-1, 1\}$. Let $u_i = \mathrm{val}(\mathcal{H}_i)$, $s_i = |\mathcal{H}_i|$ and define

$$s(U) = m + \sum_{i=0}^{m} s_i,$$

which is a measure for the encoding length of U. We then search for an $1 \leq i \leq m-1$ such that one of the following two cases holds:

(i) $\zeta_i = -1$, $\zeta_{i+1} = 1$ and there is an $\ell \in \mathbb{Z}$ such that $u_i = g^\ell$ in H.

(ii) $\zeta_i = 1$, $\zeta_{i+1} = -1$ and there is an $\ell \in \mathbb{Z}$ such that $u_i = h^\ell$ in H.

Using oracle access to the compressed power problem for H we can check in polynomial time whether one of these cases holds and compute the corresponding integer ℓ. We then replace the subsequence $\mathcal{H}_{i-1} t^{\zeta_i} \mathcal{H}_i t^{\zeta_{i+1}} \mathcal{H}_{i+1}$ by a composition system $\mathcal{H}'_i$ where $\mathrm{val}(\mathcal{H}'_i)$ is $u_{i-1} h^\ell u_{i+1}$ in case (i) and $u_{i-1} g^\ell u_{i+1}$ in case (ii). Let U' be the resulting sequence. It remains to bound $s(U')$. For this we have to bound the size of the composition system $\mathcal{H}'_i$. Assume that $\zeta_i = -1$, $\zeta_{i+1} = 1$, and $u_i = g^\ell$ in H (the case where $\zeta_i = 1$, $\zeta_{i+1} = -1$ and $u_i = h^\ell$ in H is analogous). It suffices to show that h^ℓ can be produced by a composition system $\mathcal{H}''_i$ of size $s_i + \mathcal{O}(1)$. Then we can easily bound the size of $\mathcal{H}'_i$ by $s_{i-1} + s_i + s_{i+1} + \mathcal{O}(1)$, which yields $s(U') \leq s(U) + \mathcal{O}(1)$. This shows that every sequence V that occurs during the Britton reduction satisfies $S(V) \leq S(W) + \mathcal{O}(n)$ (recall that W is the initial sequence and that the number of Britton reductions is bounded by $n/2$).

Fix the constant δ such that for every $g' \in \langle g \rangle$ the unique (since g has infinite order) $z \in \mathbb{Z}$ with $g' = g^z$ satisfies $|z| \leq \delta \cdot |g'|_\Sigma$. Hence, we have $|\ell| \leq \delta \cdot |u_i|$.

W.l.o.g. we can assume that $\delta \in \mathbb{N}$. The variables of $\mathcal{H}_i''$ are the variables of $\mathcal{H}_i$ plus two new variables A_h and S_i'. Define a morphism η by $\eta(a) = A_h$ for all $a \in \Sigma$ and $\eta(A) = A$ for every variable A of $\mathcal{H}_i$. We define the right-hand side mapping ρ_i'' of $\mathcal{H}_i''$ as follows:

- $\rho_i''(A_h) = h$ if $\ell \geq 0$ and $\rho_i''(A_h) = h^{-1}$ if $\ell < 0$ (here, we identify h and h^{-1} with words over the alphabet Σ that represent the group elements h and h^{-1}, respectively),
- $\rho_i''(S_i') = (S_i^\delta)[1 : |\ell| \cdot |h|]$, and
- $\rho_i''(A) = \eta(\rho_i(A))$ for all variables A of $\mathcal{H}_i$.

Note that S_i^δ derives to $h^{\delta \cdot |u_i|}$ if $\ell \geq 0$ and to $h^{-\delta \cdot |u_i|}$ if $\ell < 0$. Since $|\ell| \leq \delta \cdot |u_i|$, $(S_i^\delta)[1 : |\ell| \cdot |h|]$ derives to h^ℓ. The start variable of $\mathcal{H}_i''$ is S_i'. The size of $\mathcal{H}_i''$ is $s_i + |h| + \delta = s_i + \mathcal{O}(1)$, since $|h|$ and δ are constants. This concludes the proof of the theorem. $\square$

A subgroup of a hyperbolic group is undistorted if and only if it is quasiconvex [33, Lemma 1.6]. That cyclic subgroups in hyperbolic groups are quasiconvex was shown by Gromov [18, Corollary 8.1.D]. Hence, cyclic subgroups of a hyperbolic group are undistorted. Together with Theorems 4.1 and 5.1 we get:

Corollary 5.2. *Let H be a hyperbolic group and let $g, h \in H$ have the same order. Then the word problem for $\langle H, t \mid t^{-1}gt = h \rangle$ can be solved in polynomial time.*

6. Generalization to graph of groups

We can slightly generalize Corollary 5.2. For this we need the definition of a graph of groups and its fundamental group; a detailed introduction can be found in [43].

By a graph Y, we mean a graph in the sense of Serre [43]. So Y consists of a set V of vertices, a set E of edges, a function $\alpha \colon E \to V$ selecting the initial vertex of an edge, a function $\omega \colon E \to V$ selecting the terminal vertex of an edge, and a fixed-point-free involution on E written $e \mapsto e^{-1}$ (thus, $(e^{-1})^{-1} = e$ and $e \neq e^{-1}$ for all edges e) such that $\alpha(e) = \omega(e^{-1})$ for all $e \in E$. A path in Y is a sequence of edges $e_1 e_2 \cdots e_k$ such that $\omega(e_i) = \alpha(e_{i+1})$ for all $1 \leq i < k$. This path starts in $\alpha(e_1)$ and ends in $\omega(e_k)$. The graph Y is connected if for all $v, v' \in V$ there is a path starting in v and ending in v'. The involution $e \mapsto e^{-1}$ extends to paths in the natural way: $(e_1 e_2 \cdots e_k)^{-1} = e_k^{-1} \cdots e_2^{-1} e_1^{-1}$.

A *graph of groups* (G, Y) consists of a connected graph $Y = (V, E)$ and

(i) for each vertex $v \in V$, a group G_v,

(ii) for each edge $e \in E$, a group G_e such that $G_e = G_{e^{-1}}$,

(ii) for each edge $e \in E$, monomorphisms $\alpha_e \colon G_e \to G_{\alpha(e)}$ and $\omega_e \colon G_e \to G_{\omega(e)}$ such that $\alpha_e = \omega_{e^{-1}}$ for all $e \in E$.

We assume that the groups G_v intersect only in the identity, and that they are disjoint from the edge set E. For each $v \in V$, let $\langle \Sigma_v \mid R_v \rangle$ be a presentation for G_v with pairwise disjoint generating sets Σ_v. Let Δ be a set containing exactly one edge from each set $\{e, e^{-1}\}$. We identify E and $\Delta \cup \Delta^{-1}$ when convenient. Let Σ be the (disjoint) union of all the sets Σ_v and Δ. We define a group $F(G,Y)$ by the presentation

$$F(G,Y) = \langle \Sigma \mid R_v \ (v \in V),\ e^{-1}\alpha_e(g)e = \omega_e(g)\ (e \in E, g \in G_e)\rangle.$$

Fix a vertex $v_0 \in V$. A word in $w \in (\Sigma \cup \Sigma^{-1})^*$ is of *cycle type at* v_0 if it is of the form $w = w_0 e_1 w_1 e_2 w_2 \dots e_n w_n$ where:

(i) $e_i \in E$ for all $1 \le i \le n$,

(ii) $e_1 \cdots e_n$ is a path in Y starting and ending at v_0,

(iii) $w_0 \in (\Sigma_{v_0} \cup \Sigma_{v_0}^{-1})^*$, and

(iv) for $1 \le i \le n$, $w_i \in (\Sigma_{\omega(e_i)} \cup \Sigma_{\omega(e_i)}^{-1})^*$.

The images in $F(G,Y)$ of the words of cycle type at v_0 form a subgroup $\pi_1(G,Y,v_0)$ of $F(G,Y)$, called the *fundamental group of* (G,Y) *at* v_0. Since Y is connected, one can show that (up to isomorphism) $\pi_1(G,Y,v_0)$ is independent of the choice of vertex v_0; hence we simply write $\pi_1(G,Y)$. It can be also defined via a spanning tree of Y, but we do not need this. An HNN-extension $\langle H, t \mid t^{-1}at = \varphi(a)\,(a \in A)\rangle$ can be obtained as the fundamental group of a graph of groups (G,Y), where Y consists of a single vertex v, a loop e and its inverse edge e^{-1}, and $G_v = H$, $G_e = A$. Similarly, amalgamated free products are special cases of fundamental groups.

The word problem for the fundamental group $\pi_1(G,Y)$ can be solved using a generalized form of Britton reduction [14]. This consists of applying the rewriting steps $e^{-1}\alpha_e(g)e \to \omega_e(g)$ and $e\omega_e(g)e^{-1} \to \alpha_e(g)$ for $e \in E$, $g \in G_e$ as long as possible. The proof of the following theorem is completely analogous to the proof of Theorem 5.1.

Theorem 6.1. *Let (G,Y) be a graph of groups such that $Y = (V,E)$ is finite and every edge group G_e, $e \in E$, is cyclic and the image $\alpha_e(G_e)$ is undistorted in $G_{\alpha(e)}$.*[2] *Then the semi-compressed word problem for the fundamental group $\pi_1(G,Y)$ is polynomial time Turing-reducible to the compressed power problems for the vertex groups G_v, $v \in V$.*

Corollary 6.2. *Let G be a fundamental group of a graph of groups such that all vertex groups are hyperbolic and all edge groups are cyclic. Then the word problem for G can be solved in polynomial time.*

7. Future work

There is no hope to generalize Corollary 5.2 to the case of arbitrary finitely generated associated subgroups (there exists a finitely generated subgroup A of a hyperbolic group G such that the membership problem for A is undecidable [41]). On the other hand, it is known that the membership problem for quasi-convex subgroups of hyperbolic groups is decidable. What is the complexity of

[2] Then also $\omega_e(G_e)$ is undistorted in $G_{\omega(e)}$.

the word problem for an HNN-extension of a hyperbolic group H with finitely generated quasiconvex associated subgroups? Even for the case where H is free (where all subgroups are quasiconvex) the existence of a polynomial time algorithm is not clear (this problem was discussed in the introduction).

The best known complexity bound for the word problem of a hyperbolic group is LogCFL [26] , which is contained in the circuit complexity class NC^2. This leads to the question whether the complexity bound in Corollary 5.2 can be improved to NC. Also the complexity of the compressed word problem for an HNN-extension of a hyperbolic group H with cyclic associated subgroups is open (even in the case where the base group H is free). Recall that the compressed word problem for a hyperbolic group can be solved in polynomial time [24].

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