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Exercise 1

Task 1
Prove or disprove the following statements:

(a) $n \log n \in \mathcal{O}(\sqrt{n} \cdot n)$

Solution
True: We know that $\log(n) \leq \sqrt{n}$ for all $n \in \mathbb{N}$ with $n \geq 1$, so $n \log(n) \leq \sqrt{n} \cdot n$. Therefore, $n \log(n) \in \mathcal{O}(\sqrt{n} \cdot n)$ since for $c = 1$ and for all $n \geq n_0 = 1$ it holds that $n \log(n) \leq c \cdot \sqrt{n} \cdot n$.

(b) $n^{k+1} \in \mathcal{O}(n^k)$, $k \in \mathbb{N}$

Solution
False: Let $n_0 \in \mathbb{N}$, $c > 0$ and assume that for all $n \geq n_0$ it holds that $n^{k+1} \leq c \cdot n^k$, especially for $n = c + n_0 + 1 \geq n_0$ we have that $(c + n_0 + 1)^{k+1} \leq c \cdot (c + n_0 + 1)^k$. It follows that $c + n_0 + 1 \leq c$ which is a contradiction.

(c) $n^n \in \Omega(5^n)$

Solution
True: We have $n^n \in \Omega(5^n) \Leftrightarrow 5^n \in \mathcal{O}(n^n)$. We choose $c = 1$ and $n_0 = 5$, then clearly we have $5^n \leq c \cdot 5^n \leq c \cdot n^n$ for all $n \geq n_0$.

(d) $n - \log n \in o(n)$

Solution
False: Let $c = \frac{1}{2}$ and let $n_0 \in \mathbb{N}$. Then $n - \log(n) \leq cn$ means $n \leq 2\log(n)$, which is false for any $n \geq 1$.

(e) $k + \ell \cdot (-1)^n \in \Theta(1)$, $k, \ell \in \mathbb{N}$, $k > \ell$

Solution
True: We have $0 < k - \ell \leq k + \ell(-1)^n \leq k + \ell$ for any $n \in \mathbb{N}$. Hence $k + \ell(-1)^n \leq c \cdot 1$ with $c = k + \ell$, which implies $k + \ell(-1)^n \in \mathcal{O}(1)$, and $1 \leq c \cdot (k + \ell(-1)^n)$ with $c = 1$, which implies $1 \in \mathcal{O}(k + \ell(-1)^n)$.

(f) $n^3 \in \omega(n^2 \sin(n!))$

Solution
True: We have $n^3 \in \omega(n^2 \sin(n!)) \Leftrightarrow n^2 \sin(n!) \in o(n^3)$. Since $|\sin(x)| \leq 1\, \forall x \in \mathbb{N}$, we have $n^2 \sin(n!) \leq n^2 \leq c \cdot n^3$. For any $c > 0$ the inequality $n^2 \leq c \cdot n^3$ indeed holds, if n_0 is large enough: We have $1 \leq c \cdot n$ (assuming $n > 0$), which means we can pick $n_0 = \lceil 1/c \rceil$.

Alternatively we can use lim (or lim sup or lim inf) to prove these statements. For instance

$$\lim_{n \to \infty} \frac{n - \log(n)}{n} = 1 > 0,$$

hence we know $n - \log(n) \notin o(n)$.

Task 2
Let $f \colon \mathbb{N} \to \mathbb{N}$ with $f(n) \in \Theta(n)$. Prove or disprove the following statements:

(a) $f(n)^k \in \Theta(n^k)$ for all $k \in \mathbb{N}$, $k \geq 1$

Solution
True: From $f(n) \in \Theta(n)$ it follows that:

(1) $f(n) \in \mathcal{O}(n)$, so there exist $n_0 \in \mathbb{N}$, $c > 0$ such that for all $n \geq n_0$ we have $f(n) \leq c \cdot n$, which implies that $f(n)^k \leq (c \cdot n)^k = c^k \cdot n^k$, therefore $f(n)^k \in \mathcal{O}(n^k)$.

(2) $n \in \mathcal{O}(f(n))$, so there exist $n_0 \in \mathbb{N}, c > 0$ such that for all $n \geq n_0$ we have $n \leq c \cdot f(n)$, which implies that $n^k \leq (c \cdot f(n))^k = c^k \cdot f(n)^k$, therefore $n^k \in \mathcal{O}(f(n)^k)$.

We therefore obtain that $f(n)^k \in \Theta(n^k)$.

(b) $3^{f(n)} \in \Theta(3^n)$

Solution
False: Let $f(n) = 3n$, so $f(n) \in \Theta(n)$. Assume that the statement is true so we have $3^{3n} = 9^n \in \Theta(3^n)$, so $9^n \in \mathcal{O}(3^n)$. This means there exist $n_0 \in \mathbb{N}$, $c > 0$ such that for every $n \geq n_0$ we have $9^n \leq c \cdot 3^n$. This implies $(\frac{9}{3})^n \leq c$. But since 3^n is not a bouned function, this yields a contradiction.

Task 3
Use the Master Theorem to determine the asymptotic growth of the following functions:

(a) $T_1(n) = 99 \cdot T_1\left(\frac{n}{100}\right) + 24n$

Solution
$a = 99$, $b = 100$, $c = 1$. Since $a = 99 < 100 = b^c$, case 1 applies:

$$T_1(n) \in \Theta(n^c) = \Theta(n).$$

(b) $T_2(n) = 10 \cdot T_2\left(\frac{n}{3}\right) + 1000n^2$

Solution
$a = 10$, $b = 3$, $c = 2$. Since $a = 10 > 9 = b^c$, case 3 applies:

$$T_2(n) \in \Theta\left(n^{\frac{\log(a)}{\log(b)}}\right) = \Theta\left(n^{\frac{\log(10)}{\log(3)}}\right).$$

(c) $T_3(n) = 16 \cdot T_3\left(\frac{n}{2}\right) + n^4$

Solution
$a = 16, b = 2, c = 4$. Since $a = 16 = 2^4 = b^c$, case 2 applies:

$$T_3(n) \in \Theta\left(n^c \log(n)\right) = \Theta\left(n^4 \log(n)\right).$$

(d) $T_4(n) = 16 \cdot T_4\left(\frac{n}{2}\right) + n^3$

Solution
$a = 16, b = 2, c = 3$. Since $a = 16 > 8 = b^c$, case 3 applies:

$$T_4(n) \in \Theta\left(n^{\frac{\log(a)}{\log(b)}}\right) = \Theta\left(n^{\frac{\log(16)}{\log(2)}}\right) = \Theta\left(n^4\right).$$

Task 4
Give a recursive equation for the running time of the algorithm for f. Use the Master Theorem I to compute the running time of the algorithm.

function $f(n : \text{integer})$
  let m be the smallest number of the form 3^k with $3^k \geq n$;
  if $m = 1$ **then** **print**(goodbye)
  else
    for $i = 1$ to m^2 **do**
      print(hello)
    endfor
    $m := m/3$;
    for $i = 1$ to 8 **do**
      $f(m)$
    endfor
  endif
endfunction

Solution
The recursive function is $T(n) = 8 \cdot T(n/3) + \mathcal{O}(n^2)$.
We have $a = 8$, $b = 3$ and $c = 2$. Since $a = 8 < 3^2 = b^c$, the Master theorem I yields $T(n) \in \Theta(n^2)$.