Exercise 5

Task 1
Does the algorithm "Median of the Medians" run in linear time, if one uses blocks of three or blocks of nine?

Solution

- blocks of 3: $T(n) \leq T(\frac{n}{3}) + T(\frac{2n}{3}) + c \cdot n$

  The number of comparisons $T(\frac{2n}{3})$ (recursive step) is obtained like on slide 101.

  We cannot use Master Theorem II, so it is not clear, whether $T(n) \in \mathcal{O}(n)$ or not.

- blocks of 9: $T(n) \leq T(\frac{n}{9}) + T(\frac{13n}{18}) + c \cdot n$

  The number of comparisons $T(\frac{13n}{18})$ (recursive step) is obtained like on slide 101.

  Master Theorem II implies $T(n) \in \mathcal{O}(n)$, since $(\frac{1}{9} + \frac{13}{18}) < 1$.

Task 2
Which of the following pairs is a subset system, respectively matroid?

(a) $(\{1,2,3\}, \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2,3\}\})$

(b) $(\{1,2,3\}, \{\emptyset, \{1\}, \{2\}, \{3\}, \{2,3\}\})$

(c) (E, U), where E is a finite set and $U = \{A \subseteq E \mid |A| \leq k\}$ for a $k \in \mathbb{N}$.

(d) (E, U), where E is a finite subset of a vector space (for instance $\mathbb{R}^2$) and U consists of all linearly independent subsets of E.

Solution
Let E be a finite set and $U \subseteq 2^E$.
A pair (E, U) is a subset system, if $\emptyset \in U$ and $A \subseteq B \in U$ implies $A \in U$.
A subset system (E, U) is a matroid, if for all $A, B \in U$ with $|A| < |B|$ there is an element $x \in B \setminus A$ such that $A \cup \{x\} \in U$.

(a) This is not a subset system, because $\{1,2,3\} \in U$ but $\{1,2\} \notin U$.

(b) This is a subset system. The exchange property for $|\emptyset| < |A|$ for all $A \in U$ is trivial, so we have three cases to check, since $|\{1\}|, |\{2\}|, |\{3\}| < |\{2,3\}|$. For $\{1\}$ and $\{2,3\}$ there is no $x \in \{2,3\} \setminus \{1\} = \{2,3\}$ such that $\{1\} \cup \{x\} \in U$, since $\{1,2\} \notin U$ and $\{1,3\} \notin U$. Therefore, this is not a matroid.

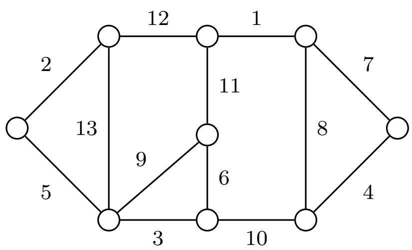
(c) This is a subset system:

- $\emptyset \in U$ because $\emptyset \subseteq E$ and $|\emptyset| = 0 \leq k$.
- Let $B \in U$, so $B \subseteq E$ and $|B| \leq k$. Let $A \subseteq B \subseteq E$. Then $|A| \leq |B|$, hence $|A| \leq k$ and therefore $A \in U$.

This is a matroid: Let $A, B \in U$ with $|A| < |B|$. Since $|B| \leq k$ we have $|A| < k$, so for every $x \in E$ it holds by definition that $A \cup \{x\} \in U$. Choose any $y \in B \setminus A \neq \emptyset$, hence $A \cup \{y\} \in U$.

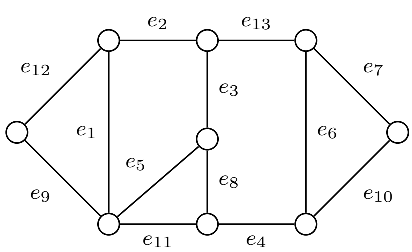
(d) This is a subset system, since $\emptyset$ is a linearly indipendent set and subsets of linearly indipendent sets are linearly indipendent. It is also a matriod: The exchange property follows from the exchange lemma of Steinitz (linear algebra).

Task 3
Compute a spanning subtree of maximal weight using Kruskal's algorithm for the following graph:

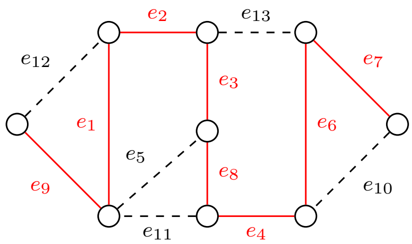


How does the result change, when you want to compute a spanning subtree of minimal weight?

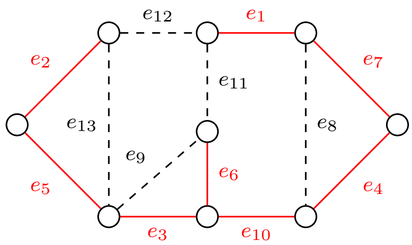
Solution
We first sort the edges by their weights in decreasing order. To illustrate better what it yields, we show the graph one more time:



Kruskal's algorithm now takes greedily any heavy edge into the set F, such that (V, F) has no cycles (V is just the vertex set from the original graph).
In the end the spanning subtree has the following edges: $F = \{e_1, e_2, e_3, e_4, e_6, e_7, e_8, e_9\}$.



To compute a spanning subtree of minimal weight, we need to sort the edges in reversed order. After going through all the steps, we obtain (details: see exercise session)



Hence, the graph is (V, F'), where $F' = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_{10}\}$.

Task 4

(a) Show that for each tree $T = (V, E)$ with $|V| > 0$ we have $|E| = |V| - 1$.

(b) Show that every finite connected graph has a spanning subtree.

Solution

(a) We do an induction on $|V|$. In case $|V| = 1$ it is clear that $|E| = 0$. Now let $|V| > 1$. Since T has no cycles, there is a leaf in T, meaning there is a $v \in V$ with $|v_E| = 1$, where $v_E = \{\{v, u\} \in V^2 \mid \{v, u\} \in E\}$. So $|v_E| = 1$ means that v borders only one edge, which means that the node u with $\{v, u\} \in E$ is the parent node of v. Let $T' = (V', E')$ with $V' \setminus \{v\}$ and $E' = E \setminus v_E$. This is a tree, since it is connected, because T is connected and v is a leaf, and it also has no cycles, since $E' \subseteq E$ and T has no cycles. Furthermore, $|V'| = |V| - 1$, so by induction hypothesis we obtain $|E'| = |V'| - 1$. We have now proven that $|E| = |E'| + 1 = |V'| - 1 + 1 = |V'| = |V| - 1$.

(b) Let $G = (V, E)$ be a connected graph. If G is a tree, G is a spanning tree of G. Otherwise, choose an edge $e \in E$ that is on a cycle in G and let $E' = E \setminus \{e\}$. Now $G' = (V, E')$ is still connected and $E' \subset E$. We set $G = G'$ and iterate the above step. This algorithm terminates because G is finite and we remove one edge in each step. Repeatedly removing edges on cycles in a finite graph eventually leads to a graph that has no cycles and is therefore a (spanning) subtree.