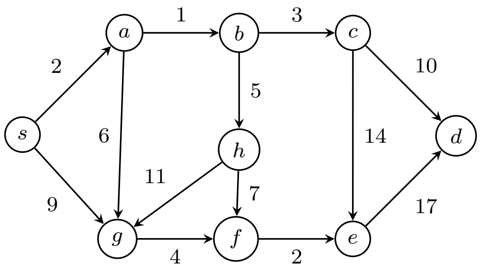
Exercise 6

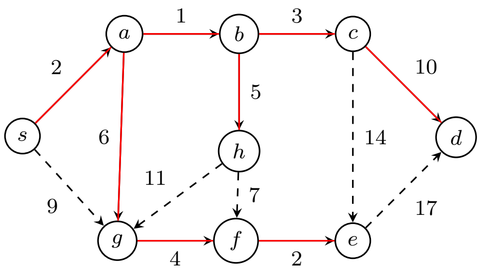
Task 1

Use Dijkstra's algorithm to compute all shortest paths starting at node s. Show the values of the program variables B, R, U, p, D after each iteration of the main **while**-loop of Dijkstra's algorithm.



Solution

For Dijkstra's algorithm it is useful to draw a table (with values D and p) and indicate the tree nodes (B), the boundary nodes (R) and the unknown nodes (U) within this table. The latter ones have distance ∞ to the tree nodes (where distance is **bold**), since they cannot be reached in one step. The boundary nodes are just written with normal (non bold) distance. The shortest paths are resulting in a tree, highlighted in red.



Node	s	a	b	c	d	e	f	g	h
Step 0	0	∞	∞	∞	∞	∞	∞	∞	∞
$p(x)$	nil	nil	nil	nil	nil	nil	nil	nil	nil
Step 1	**0**	2	∞	∞	∞	∞	∞	9	∞
$p(x)$	nil	s	nil	nil	nil	nil	nil	s	nil
Step 2	**0**	**2**	3	∞	∞	∞	∞	8	∞
$p(x)$	nil	s	a	nil	nil	nil	nil	a	nil
Step 3	**0**	**2**	**3**	6	∞	∞	∞	8	8
$p(x)$	nil	s	a	b	nil	nil	nil	a	b
Step 4	**0**	**2**	**3**	**6**	16	20	∞	8	8
$p(x)$	nil	s	a	b	c	c	nil	a	b
Step 5	**0**	**2**	**3**	**6**	16	20	12	**8**	8
$p(x)$	nil	s	a	b	c	c	g	a	b
Step 6	**0**	**2**	**3**	**6**	16	20	12	**8**	**8**
$p(x)$	nil	s	a	b	c	c	g	a	b
Step 7	**0**	**2**	**3**	**6**	16	14	**12**	**8**	**8**
$p(x)$	nil	s	a	b	c	f	g	a	b
Step 8	**0**	**2**	**3**	**6**	16	**14**	**12**	**8**	**8**
$p(x)$	nil	s	a	b	c	f	g	a	b
Step 9	**0**	**2**	**3**	**6**	**16**	**14**	**12**	**8**	**8**
$p(x)$	nil	s	a	b	c	f	g	a	b

Task 2

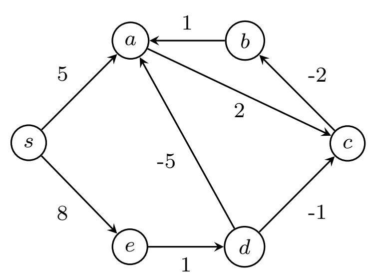
In this task we want to consider directed graphs with possible negative-weighted edges. In many cases Dijkstra's algorithm will not yield a correct result.

(a) Assuming that all nodes are reachable from the source node s, what other necessary condition has to be assumed to guarantee the existence of shortest paths to each node?

Solution

Since all nodes are reachable, we need the extra condition that the graph has no negative cycles (such as a cycle $a \to b \to c \to a$ labeled with $2, 1$ and -4). Otherwise we can have infinitely short paths (weight $-\infty$).

(b) Consider the following graph:



Why does Dijkstra's algorithm not work in this case (source node s), even though shortest paths exist to each node of the graph?

Solution
In step 1 node a will be visited (distance to s is 5). But the shortest path is $s \to e \to d \to a$ with total weight / distance $8 + 1 - 5 = 4 < 5$. Dijkstra's algorithm cannot guess, if negative edges with a very small weight will appear a few steps ahead.

(c) Modify Dijkstra's algorithm, such that for every weighted directed graph with source node s one always obtains the shortest path to each node (assuming their existence). What is the running time of your algorithm?

Solution
Let $G = (V, E, \gamma)$ be the weighted graph. Let $n = |V|$ and $e = |E|$. The easiest way is to do $n - 1$ iterations (longest minimal weighted path length) of the following steps: In step i, check if the distance to s until step $i - 1$ to each node is $< \infty$, then update all values in the table for all outgoing edges from each node to any other. With other words, you can visit each node multiple times. This algorithm is essentially known as the Bellman-Ford-Algorithm.

The running time is $\mathcal{O}(e \cdot n)$, which is worse than Dijkstra's algorithm. Several optimizations can be made, such as stopping the interations, when there is no change in the table. Also there are other even more effective algorithms, but obviously the running time with possible negative weights is still worse.

(d) Test your algorithm on the example of part b. The distance variable is sufficient.

Solution
We obtain the following table for the distances (max. 5 steps):

Node	s	a	b	c	d	e
Step 0	0	∞	∞	∞	∞	∞
Step 1	0	5	∞	∞	∞	8
Step 2	0	5	∞	7	9	8
Step 3	0	4	5	7	9	8
Step 4	0	4	5	6	9	8
Step 5	0	4	4	6	9	8

Task 3
Show Theorem 17 from the lecture (slide 155) : For all $k \geq 0$ we have

$$F_k = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^{k+1} - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^{k+1}.$$

Here we use $F_0 = F_1 = 1$ (there are other conventions).

Solution

Let $x^2 = x + 1$. The two solutions to this equation are $r := \frac{1+\sqrt{5}}{2}$ and $s := \frac{1-\sqrt{5}}{2}$, so we know that $r^2 = r + 1$ and $s^2 = s + 1$.

For $k = 0$ we have

$$\frac{1}{\sqrt{5}} r^1 - \frac{1}{\sqrt{5}} s^1 = \frac{1}{\sqrt{5}} (r - s) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} \right) = 1 = F_0$$

For $k = 1$ we have

$$\frac{1}{\sqrt{5}} r^2 - \frac{1}{\sqrt{5}} s^2 = \frac{1}{\sqrt{5}} (r^2 - s^2) = \frac{1}{\sqrt{5}} \left((r+1) - (s+1) \right) = \frac{1}{\sqrt{5}} (r - s) = 1 = F_1$$

Assume the statement is already true for k. Now we prove it for $k + 1$:

$$\begin{aligned}
F_{k+1} &= F_k + F_{k-1} \\
&= \frac{1}{\sqrt{5}} r^{k+1} - \frac{1}{\sqrt{5}} s^{k+1} + \frac{1}{\sqrt{5}} r^k - \frac{1}{\sqrt{5}} s^k \\
&= \frac{1}{\sqrt{5}} \left(r^{k+1} - s^{k+1} + r^k - s^k \right) \\
&= \frac{1}{\sqrt{5}} \left(r^k (r+1) - s^k (s+1) \right) \\
&= \frac{1}{\sqrt{5}} \left(r^{k+2} - s^{k+2} \right) \\
&= \frac{1}{\sqrt{5}} r^{k+2} - \frac{1}{\sqrt{5}} s^{k+2}
\end{aligned}$$