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Exercise 5

Task 1
Let $(\mathbb{Z}, +, \cdot)$ be a structure, where

- $\mathbb{Z}$ denotes the universe of the structure,
- $+$ denotes a binary function symbol interpreted as the addition of integers, and
- $\cdot$ denotes a binary function symbol interpreted as the multiplication of integers.

Show that $\mathrm{Th}(\mathbb{Z}, +, \cdot)$ is undecidable.
Hint: Apply Lagrange's four-square theorem:

Theorem 1 (Lagrange's four-square theorem)
Every natural number can be represented as the sum of four integer squares, that is, for every $x \in \mathbb{N}$, there are integers $x_1, x_2, x_3, x_4 \in \mathbb{Z}$, such that $x = x_1^2 + x_2^2 + x_3^2 + x_4^2$.

Task 2
Consider the structure $(\mathbb{N}, +, \cdot, s, 0)$. Use Gödel's β-function in order to formalize the following statements in predicate logic:

(a) $x^y = z$ (use free variables x, y and z),

(b) Fermat's Last Theorem,

(c) Collatz conjecture.