Exercise 1

Task 1
Find a model for each of the following formulas of predicate logic, and structures in which the formulas evaluate to false.

(a) $\exists x \forall y (f(f(y)) = x)$

(b) $\exists x \exists y (P(x,y) \wedge \neg P(y,x))$

(c) $\forall x \big(f(g(f(x))) \neq g(f(g(x))) \big)$

(d) $R(x) \wedge Q(y) \wedge \forall x (\neg R(x) \vee \neg Q(x))$

Solution:

(a) Model: $\mathcal{A} = (\mathbb{N}, I_{\mathcal{A}})$, with $f^{\mathcal{A}}(x) = 1$ (for every $x \in \mathbb{N}$). The formula evaluates to true in the structure $\mathcal{A}$: There exists an element $x \in \mathbb{N}$ (which is $x = 1$), such that $f^{\mathcal{A}}(f^{\mathcal{A}}(y)) = x$ for every $y \in \mathbb{N}$. Another model: $\mathcal{A}' = (\{1\}, I_{\mathcal{A}'})$, with $f^{\mathcal{A}'}(1) = 1$.

A structure in which the formula evaluates to false: $\mathcal{B} = (\mathbb{N}, I_{\mathcal{B}})$, with $f^{\mathcal{B}}(x) = x$. The formula evaluates to false in the structure $\mathcal{B}$: There is no element $x \in \mathbb{N}$, such that for every $y \in \mathbb{N}$, we have $f^{\mathcal{B}}(f^{\mathcal{B}}(y)) = x$. Another example: $\mathcal{B}' = (\{0,1\}, I_{\mathcal{B}'})$, with $f^{\mathcal{B}'}(x) = 1 - x$.

(b) Model: $\mathcal{A} = (\mathbb{N}, I_{\mathcal{A}})$, with $P^{\mathcal{A}} = \{(x,y) \in \mathbb{N} \times \mathbb{N} \mid x < y\}$. The formula evaluates to true in the structure $\mathcal{A}$: There are elements $x, y \in \mathbb{N}$ (for example $x = 1, y = 2$) which satisfy $x < y$ (and thus, $(x,y) \in P^{\mathcal{A}}$), but which do not satisfy $y < x$ (such that $(y,x) \notin P^{\mathcal{A}}$). Another example: $\mathcal{A}' = (\{0,1\}, I_{\mathcal{A}'})$, with $P^{\mathcal{A}'} = \{(1,0), (0,0)\}$.

A structure in which the formula evaluates to false: $\mathcal{B} = (\mathbb{N}, I_{\mathcal{B}})$, with $P^{\mathcal{B}} = \emptyset$. The formula evaluates to false in the structure $\mathcal{B}$, as the relation $P^{\mathcal{B}}$ is empty: Hence there is no pair (x,y) with $(x,y) \in P^{\mathcal{B}}$. Another example: $\mathcal{B}' = (\{1\}, I_{\mathcal{B}'})$, with $P^{\mathcal{B}'} = \{(1,1)\}$.

(c) Model: $\mathcal{A} = (\mathbb{N}, I_{\mathcal{A}})$ with $f^{\mathcal{A}}(x) = 1$ and $g^{\mathcal{A}}(x) = 2$ for every $x \in \mathbb{N}$. The formula $\mathcal{A}$ evaluates to true in the structure, as $f^{\mathcal{A}}(g^{\mathcal{A}}(f^{\mathcal{A}}(x))) = 1$ for every $x \in \mathbb{N}$ and $g^{\mathcal{A}}(f^{\mathcal{A}}(g^{\mathcal{A}}(x))) = 2$ for every $x \in \mathbb{N}$. Another example: $\mathcal{A}' = (\mathbb{Z}, I_{\mathcal{A}'})$ with $f^{\mathcal{A}'}(x) = x - 1$ and $g^{\mathcal{A}'}(x) = x + 1$.

A structure in which the formula evaluates to false: $\mathcal{B} = (\mathbb{N}, I_{\mathcal{B}})$, with $f^{\mathcal{B}}(x) = x$ and $g^{\mathcal{B}}(x) = x$ for every $x \in \mathbb{N}$. The formula evaluates to false in this structure $\mathcal{B}$, as $f^{\mathcal{B}} = g^{\mathcal{B}}$ and hence $f^{\mathcal{B}}(g^{\mathcal{B}}(f^{\mathcal{B}}(x))) = g^{\mathcal{B}}(f^{\mathcal{B}}(g^{\mathcal{B}}(x)))$ for every $x \in \mathbb{N}$. Another example: $\mathcal{B}' = (\mathbb{R}, I_{\mathcal{B}'})$, wobei $f^{\mathcal{B}'}(x) = x^2$ und $g^{\mathcal{B}'}(x) = x^3$ (for $x = 1$, we have $f^{\mathcal{B}}(g^{\mathcal{B}}(f^{\mathcal{B}}(x))) = g^{\mathcal{B}}(f^{\mathcal{B}}(g^{\mathcal{B}}(x))))$.

(d) Model: $\mathcal{A} = (\mathbb{N}, I_{\mathcal{A}})$, with $x^{\mathcal{A}} = 2$, $y^{\mathcal{A}} = 3$, $R^{\mathcal{A}} = \{2x \mid x \in \mathbb{N}\}$ und $Q^{\mathcal{A}} = \mathbb{N} \setminus R^{\mathcal{A}}$. The formula evaluates to true in the structure: $R^{\mathcal{A}}$ is the set of even numbers, $Q^{\mathcal{A}}$ is the set of odd numbers. We find that $x^{\mathcal{A}} = 2$ is even and $y^{\mathcal{A}} = 3$ is odd. Furthermore, every $x \in \mathbb{N}$ is not even or not odd. Another model: $\mathcal{A}' = (\{0, 1\}, I_{\mathcal{A}'})$ with $x^{\mathcal{A}'} = 0$, $y^{\mathcal{A}'} = 1$, $R^{\mathcal{A}'} = \{0\}$ und $Q^{\mathcal{A}'} = \{1\}$.

A structure in which the formula evaluates to false: $\mathcal{B} = (\mathbb{N}, I_{\mathcal{B}})$, with $x^{\mathcal{B}} = y^{\mathcal{B}} = 1$ und $R^{\mathcal{B}} = Q^{\mathcal{B}} = \mathbb{N}$. The formula evaluates to false in the structure, as for every $x \in \mathbb{N}$, $R^{\mathcal{B}}(x)$ and $Q^{\mathcal{B}}(x)$ evaluate to true. Another example: $\mathcal{B}' = (\{0, 1\}, I_{\mathcal{B}'})$, with $x^{\mathcal{B}'} = y^{\mathcal{B}'} = 1$ and $R^{\mathcal{B}'} = Q^{\mathcal{B}'} = \{0\}$.

Task 2

Let f denote a binary function symbol and let R be a unary predicate symbol. Consider the following structures:

- $\mathcal{A}_1 = (\mathbb{N}, I_{\mathcal{A}_1})$, with $f^{\mathcal{A}_1}(x, y) = x \cdot y,\ R^{\mathcal{A}_1} = \{n \in \mathbb{N} \mid n \text{ is prime}\}$
- $\mathcal{A}_2 = (\mathbb{R}, I_{\mathcal{A}_2})$, with $f^{\mathcal{A}_2}(x, y) = x - 2y,\ R^{\mathcal{A}_2} = \{x \in \mathbb{R} \mid x \leq 0\}$

Do the following formulas evaluate to true in these structures?

(a) $\forall x(R(x) \lor R(f(x, x)))$

(b) $\forall x \exists y R(f(x, y))$

(c) $\forall x \forall y(f(x, y) = f(y, x))$

Solution:

(a) The structure $\mathcal{A}_1$ is not a model for this formula: The number $4 \in \mathbb{N}$ for example is not a prime and $f^{\mathcal{A}_1}(4, 4) = 4 \cdot 4 = 16$ is not a prime either. The structure $\mathcal{A}_2$ is a model for this formula: We have $f^{\mathcal{A}_2}(x, x) = x - 2x = -x$, and for every real number x, we find that $x \leq 0$ or $-x \leq 0$, such that $x \in R^{\mathcal{A}_2}$ or $f^{\mathcal{A}_2}(x, x) \in R^{\mathcal{A}_2}$.

(b) The structure $\mathcal{A}_1$ is not a model for this formula: For example for $x = 4$ there is no $y \in \mathbb{N}$, such that $x \cdot y$ is a prime. The structure $\mathcal{A}_2$ is a model for this formula: For every $x \in \mathbb{R}$ there is $y \in \mathbb{R}$ such that $x - 2y \leq 0$.

(c) The structure $\mathcal{A}_1$ is a model for this formula, as multiplication of natural numbers is commutative, that is, $x \cdot y = y \cdot x$ for every $x, y \in \mathbb{N}$. The structure $\mathcal{A}_2$ is not a model for this formula: For example for $x = 1$ and $y = 2$, we find that $x - 2y = -3 \neq 0 = y - 2x$.

Task 3

Let $L \subseteq \Sigma^*$ be a formal language over the alphabet Σ. Answer the following questions:

(a) How is the complement of L defined?

(b) When do we call a language L decidable, and how is the characteristic function χ_L of L defined?

(c) When do we call a language L recursively enumerable, and how is the semi-characteristic function χ'_L of L defined?

Solution:

(a) The complement of $L \subseteq \Sigma^*$ is defined as $\bar{L} = \Sigma^* \setminus L$. In other words, the complement of L contains all words $w \in \Sigma^*$, which are not contained in L.

(b) The characteristic function χ_L of a formal language L is defined as follows:

$$\chi_L(w) = \begin{cases} 1 & w \in L \\ 0 & w \notin L. \end{cases}$$

A language L is called decidable if there is an algorithm (a Turing machine, ...) with the following properties: For every $w \in \Sigma^*$, we have that

- if $w \in L$, the algorithm terminates on input w with output 1,
- if $w \notin L$, the algorithm terminates on input w with output 0.

In other words, a language L is decidable, if its characteristic function χ_L is computable.

(c) The semi-characteristic function χ'_L of L is defined as follows:

$$\chi'_L(w) = \begin{cases} 1 & w \in L \\ \text{undefined} & w \notin L. \end{cases}$$

A language L is called recursively enumerable, if there is an algorithm (a Turing machine,...) with the following properties: For every $w \in \Sigma^*$, we have that

- if $w \in L$, the algorithm terminates on input w,
- if $w \notin L$, the algorithm does not terminate on input w.

In other words, a language L is recursively enumerable, if its semi- characteristic function χ'_L is computable.

The following statements hold:

- A language L is recursively enumerable, if and only if there is a computable total function $f : \mathbb{N} \to \Sigma^*$ with $L = \{f(i) \mid i \in \mathbb{N}\}$
- A language L is decidable if and only if L and $\bar{L}$ are recursively enumerable.