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Exercise 2

Task 1
Which of the following problems are decidable, and which of them are recursively enumerable?

(a) Checking whether a formula F of predicate logic is neither valid nor unsatisfiable,

(b) checking whether a formula F of propositional logic (Aussagenlogik) is valid,

(c) checking whether a formula F of predicate logic without any existential quantifiers is satisfiable,

(d) checking whether a formula F of predicate logic without any existential quantifiers and universal quantifiers is satisfiable.

Solution:

(a) This problem is not recursively enumerable (and thus, not decidable). Let us assume that this problem is recursively enumerable (to deduce a contradiction): That is, we assume that there is an algorithm (algorithm 1) which outputs 'yes' on input F, if F is neither valid nor unsatisfiable, and does not terminate otherwise. Furthermore, we already know that the set of valid formulas of predicate logic is recursively enumerable (Corollary of Gilmore's Theorem). That is, there is an algorithm (algorithm 2), which outputs 'yes' on input F, if F is valid, and does not terminate otherwise. Combining these two algorithms gives us an algorithm (algorithm 3) to check whether a given formula F of predicate logic is satisfiable: Run algorithm 1 and algorithm 2 in parallel (both with input F). If

- algorithm 2 terminates: The formula is valid and hence satisfiable. We output 'yes'
- algorithm 1 terminates: The formula is not unsatisfiable, and hence satisfible. We output 'yes'.

Otherwise, if algorithm 1 and 2 do not terminate, F must be unsatisfiable and algorithm 3 does not terminate either. Hence, algorithm 3 terminates, if the input F is satisfiable, and does not terminate otherwise. However, the set of satisfiable formulas of predicate logic is not recursively enumerable (Corollary of Church's theorem), a contradiction.

(b) This problem is decidable. Let n be the number of variables occurring in the formula of propositional logic. In order to check whether F is valid, we have to assign truth values 0 and 1 to all variables occurring in F and check whether F evaluates to true for all possible truth assignments to the variables. In total, there are 2^n different truth assignments to the variables in F. Evaluating F with respect to these finitely many truth assignments can be done in finite time.

(c) This problem is not recursively enumerable (and hence not decidable). Every formula F of predicate logic is equivalent to a formula F' of predicate logic that does not contain any existential quantifiers (this follows from the equivalence $\exists x G \equiv \neg \forall x \neg G$, where G is an arbitrary formula of predicate logic). From the corollary of Church's Theorem, we know that the set of satisfiable formulas of predicate logic is not recursively enumerable.

(d) This problem is decidable. A formula of predicate logic without any quantifiers thus only consists of conjuctions, disjunctions and negations of atomic formulas. Let n be the number of atomic formulas occurring in F, then there are 2^n possible truth assignments to these atomic formulas. We can thus check in finite time, whether F evaluates to true for one of these possible truth assignments.

Task 2

Let $(\mathbb{N}, +, \cdot)$ be a structure, where

- $\mathbb{N}$ denotes the universe of the structure,
- $+$ und $\cdot$ are binary function symbols, interpreted as the addition and multiplication of natural numbers,
- the binary relation $=$ denotes equality of two natural numbers.

Find formulas of predicate logic for the following statements:

(a) x is a prime number (use a free variable x).

(b) z is the greatest common divisor of x and y (use free variables x, y, z).

(c) x and y are coprime (use free variables x and y).

(d) There is no largest prime number.

(e) Every number except for 1 is the product of a prime number and a natural number.

(f) Every prime number except for 2 is odd.

(g) Every even number which is greater than 2 is a sum of two prime numbers (Goldbach's conjecture).

(h) There are infinitely many prime numbers p, such that $p + 2$ is a prime number as well.

Solution:

(a) First, we define $x = 1$ for a variable x as $\forall y(x \cdot y = y)$. Then

$$\text{prim}(x) := \neg(x = 1) \wedge \forall u \forall v\, ((u \cdot v = x) \to ((u = 1) \vee (v = 1)))$$

(b) We define $x \leq y$ as $\exists z(x + z = y)$. Furthermore, we define $z|x, y$ (z divides x and y) as $\exists u \exists v((x = u \cdot z) \wedge (y = v \cdot z))$. Then

$$z = \text{gcd}(x, y) := (z|x, y) \wedge \forall u((u|x, y) \to (u \leq z)).$$

(c) $(z = 1) \wedge (z = \text{gcd}(x, y))$

(d) First, we define $x < y$ as $(x \leq y) \wedge \neg(x = y)$. Then

$$\forall x(\text{prim}(x) \to \exists y(\text{prim}(y) \wedge (x < y))).$$

(e) $\forall x(\neg(x = 1) \to \exists y \exists z(\text{prim}(y) \wedge (x = y \cdot z)))$

(f) We define $\text{odd}(x)$ (x is odd) as $\neg\exists y(x = y + y)$. Furthermore, we define $x = 2$ as $\exists y((y = 1) \wedge (x = y + y))$. Then

$$\forall x(\neg(x = 2) \to (\text{prim}(x) \to \text{odd}(x)))$$

(g) We define $\text{even}(x) = \neg\text{odd}(x)$ (x is even). Then

$$\forall x((\text{even}(x) \wedge \exists y((y = 2) \wedge (y < x))) \to \exists p \exists q(\text{prim}(p) \wedge \text{prim}(q) \wedge (x = p + q)))$$

(h) First, we define $x = y + 2$ as $\exists w(w = 2 \wedge x = y + w)$. Then

$$\forall x \exists y(\text{prim}(y) \wedge (x < y) \wedge \exists z(\text{prim}(z) \wedge (z = y + 2))))$$