Exercise 3

Task 1
Consider the table below. Fill in "✓" or "✗" in the cells, if the respective set of formulas of predicate logic is decidable/recursively enumerable. From which theorems do the respective results follow?

Set of .... formulas of predicate logic	decidable	recursively enumerable
Unsatisfiable		
Valid		
Satisfiable		
Finitely unsatisfiable		
Finitely valid		
Finitely satisfiable		

Solution:

	decidable	recursively enumerable
Unsatisfiable	✗	✓ (Corollary of Gilmore's Theorem, slide 6)
Valid	✗ (Church's Theorem)	✓ (Corollary slide 6, follows from Gilmore's Theorem)
Satisfiable	✗	✗ (Corollary of Church's Theorem, slide 7)
Finitely unsatisfiable	✗	✗ (Corollary slide 26)
Finitely valid	✗	✗ (Corollary slide 26)
Finitely satisfiable	✗(Trachtenbrot's Theorem)	✓ (Lemma slide 26)

The marks "✗" for decidability of the set of satisfiable formulas, the set of finitely unsatisfiable formulas and the set of finitely valid formulas follow from the fact that a set that is not recursively enumerable cannot be decidable. The mark "✗" for decidability of the set of unsatisfiable formulas follows from the fact that its complement (the set of satisfiable formulas) is not decidable.

Task 2
Consider the following formulas of predicate logic. Which of these formulas are satisfiable? Which of these formulas are finitely satisfiable?

(a) $(\exists x P(x) \wedge \forall x \neg P(x))$, where P is a unary predicate symbol

(b) $(\forall y \exists x\ f(x) = y \wedge \exists u \exists v (f(u) = f(v) \wedge u \neq v))$, where f is a unary function symbol

(c) $((\forall x \forall y R(x, y)) \rightarrow (\exists u \exists v R(u, v)))$, where R is a binary predicate symbol

(d) $(\forall x \forall y (g(x) = g(y) \rightarrow x = y) \wedge \exists u \forall v\ g(v) \neq u)$, where g is a unary function symbol

(e) $\forall x \forall y (R(x, y, f(x, y)) \wedge \neg R(f(x, y), x, y) \wedge \neg R(x, f(x, y), y))$, where R is a 3-ary predicate symbol and f is a binary function symbol

(f) $(\forall x \neg R(x, x) \wedge \forall y \forall z ((y \neq z) \rightarrow (R(y, z) \vee R(z, y))) \wedge \forall x \forall y \forall z ((R(x, y) \wedge R(y, z)) \rightarrow R(x, z)) \wedge \forall u \exists v R(u, v))$, where R is a binary predicate symbol.

Solution:

(a) This formula is logically equivalent to $(\exists x P(x) \wedge \neg \exists x P(x))$. Hence, this formula is unsatisfiable.

(b) This formula is satisfiable, but not finitely satisfiable: The formula states that the function f is surjective, but not injective. On finite sets, surjectivity and injectivity are equivalent. Thus, there is no model with finite universe for this formula. The formula is satisfiable, a model $\mathcal{A} = (U_{\mathcal{A}}, I_{\mathcal{A}})$ for the formula with infinite universe is for example given by $U_{\mathcal{A}} = \mathbb{N}$, $f^{\mathcal{A}}(x) = \lfloor x/2 \rfloor$.

(c) This formula is satisfiable and finitely satisfiable. For example, let $\mathcal{A} = (U_{\mathcal{A}}, I_{\mathcal{A}})$ with $U_{\mathcal{A}} = \{0\}$ and $R^{\mathcal{A}} = \{(0, 0)\}$, then $\mathcal{A}$ is a model for the formula (with finite universe). The formula is even valid.

(d) This formula is satisfiable, but not finitely satisfiable. The formula states that the function f is injective, but not surjective. Hence, as in task (b), as surjectivity and injectivity on finite sets are equivalent, this formula is not finitely satisfiable. The formula is satisfiable. Take, for example, $\mathcal{A} = (U_{\mathcal{A}}, I_{\mathcal{A}})$ with $U_{\mathcal{A}} = \mathbb{R}$, $f^{\mathcal{A}}(x) = e^x$ as a model.

(e) This formula is satisfiable and finitely satisfiable, it is possible to find a model with finite universe for this formula.

(f) This formula is satisfiable, but not finitely satisfiable. The formula states that the relation R is not reflexive, total and transitive and that for each element u there exists another element v that is “greater” than u with respect to this relation. This is only possible on infinite sets.

Task 3
True or false?

(a) $\forall x \exists y\,(x = y \cdot y) \in \mathrm{Th}(\mathbb{N}, +, \cdot)$

(b) $\forall x \exists y\,(x = y + y) \in \mathrm{Th}(\mathbb{R}, +, \cdot)$

(c) $\exists x \forall y\, x < y \in \mathrm{Th}(\mathbb{N}, <)$

(d) $\forall x \exists y (P(y) \wedge (x < y) \wedge \exists z (P(z) \wedge (z = y + 2))) \in \mathrm{Th}(\mathbb{N}, +, <, P, 2)$, where P is the set of prime numbers.

Solution:

(a) False (for example, there is no natural number y such that $y \cdot y = 5$)

(b) True

(c) False (as $0 < 0$ does not hold)

(d) This is the twin prime conjecture, which is still open.