Exercise 4

Task 1

Let $\mathcal{A}$ be a structure. Show that $\text{Th}(\mathcal{A})$ is decidable if and only if $\text{Th}(\mathcal{A}_{rel})$ is decidable.

Solution:
We have to show two directions:

The direction „If $\text{Th}(\mathcal{A}_{rel})$ is decidable then $\text{Th}(\mathcal{A})$ is decidable“ was already shown in the lecture (slides 32-33).

It remains to show the other direction: „If $\text{Th}(\mathcal{A})$ is decidable then $\text{Th}(\mathcal{A}_{rel})$ is decidable“. For this, we transform a formula F (such that $\mathcal{A}_{rel}$ is suitable for F) into a formula F' (such that $\mathcal{A}$ is suitable for F'), such that $\mathcal{A}_{rel} \models F$ holds if and only if $\mathcal{A} \models F'$. Under the assumption that $\text{Th}(\mathcal{A})$ is decidable, we then obtain the following decision algorithm for $\text{Th}(\mathcal{A}_{rel})$ which checks whether $F \in \text{Th}(\mathcal{A}_{rel})$:

  (i) Construct the formula F' from F, such that $\mathcal{A}_{rel} \models F$ holds if and only if $\mathcal{A} \models F'$,

  (ii) Use the decision algorithm for $\text{Th}(\mathcal{A})$ in order to check whether $F' \in \text{Th}(\mathcal{A})$,

  (iii) Output „yes“, if $F' \in \text{Th}(\mathcal{A})$ and „no“ otherwise.

Let F be a formula, such that $\mathcal{A}_{rel}$ is suitable for F. Every atomic formula in F is of the form $R(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are variables (as F does not contain any function symbols). If R corresponds to a predicate symbol in $\mathcal{A}$, then we leave $R(x_1, \ldots, x_n)$ in F' as it is. If R corresponds to a function symbol f in $\mathcal{A}$, then we replace $R(x_1, \ldots, x_n)$ in F' by $x_n = f(x_1, \ldots, x_{n-1})$. By definition of $\mathcal{A}_{rel}$, we find that $\mathcal{A}_{rel} \models F$ holds if and only if $\mathcal{A} \models F'$.

Task 2

Let $(\mathbb{N}, +, f)$ be a structure, where

- $\mathbb{N}$ denotes the universe of the structure,

- $+$ denotes a binary function symbol interpreted as the addition of natural numbers, and

- f denotes a unary function symbol interpreted as the function $f \colon \mathbb{N} \to \mathbb{N}$ with $f(x) = x^2$.

Show that $\text{Th}(\mathbb{N}, +, f)$ is undecidable.

Solution:

Let F be a formula such that $(\mathbb{N}, +, \cdot)$ is suitable for F. We transform F into a formula F' such that $(\mathbb{N}, +, f)$ is suitable for F', and such that $F \in \mathrm{Th}(\mathbb{N}, +, \cdot)$ if and only if $F' \in \mathrm{Th}(\mathbb{N}, +, f)$. Let $\mathcal{A}_{rel} = (\mathbb{N}, P_1, P_2)$ denote the relational structure corresponding to $(\mathbb{N}, +, \cdot)$ with

$$P_1 = \{(a_1, a_2, a) \in \mathbb{N}^3 \mid a_1 + a_2 = a\} \quad \text{and}$$
$$P_2 = \{(a_1, a_2, a) \in \mathbb{N}^3 \mid a_1 \cdot a_2 = a\}.$$

Using the construction from the lecture (slide 33), we transform F into a formula F'', such that $\mathcal{A}_{rel}$ is suitable for F'' and $F \in \mathrm{Th}(\mathbb{N}, +, \cdot)$ if and only if $F'' \in \mathrm{Th}(\mathcal{A}_{rel})$. Every atomic formula in F'' is of the form $P_1(x_1, x_2, x_3)$, $P_2(x_1, x_2, x_3)$ (or $x_1 = x_2$), where x_1, x_2, x_3 are variables. We find that $z = x \cdot y$ can be defined in $(\mathbb{N}, +, f)$ as

$$z = x \cdot y \quad := \quad f(x) + f(y) + z + z = f(x + y).$$

In order to obtain F', replace every occurrence of $P_1(x_1, x_2, x)$ in F'' by $x_1 + x_2 = x$ and every occurrence of $P_2(x_1, x_2, x)$ in F'' by $f(x_1) + f(x_2) + x + x = f(x_1 + x_2)$: This yields a formula F', such that $(\mathbb{N}, +, f)$ is suitable for F' and such that $F \in \mathrm{Th}(\mathbb{N}, +, \cdot)$ if and only if $F' \in \mathrm{Th}(\mathbb{N}, +, f)$. If $\mathrm{Th}(\mathbb{N}, +, f)$ were decidable, we could obtain a decision algorithm for $\mathrm{Th}(\mathbb{N}, +, \cdot)$ by transforming a formula F (for which $(\mathbb{N}, +, \cdot)$ is suitable) into a formula F' (for which is $(\mathbb{N}, +, f)$ suitable), such that $F \in \mathrm{Th}(\mathbb{N}, +, \cdot)$ if and only if $F' \in \mathrm{Th}(\mathbb{N}, +, f)$, and then use the decision algorithm for $\mathrm{Th}(\mathbb{N}, +, f)$ to check whether $F' \in \mathrm{Th}(\mathbb{N}, +, f)$. As $\mathrm{Th}(\mathbb{N}, +, \cdot)$ is undecidable by Gödel's Theorem, $\mathrm{Th}(\mathbb{N}, +, f)$ is undecidable as well.