Exercise 5

Task 1

Let $(\mathbb{Z}, +, \cdot)$ be a structure, where

- Z denotes the universe of the structure,
- + denotes a binary function symbol interpreted as the addition of integers, and
- \cdot denotes a binary function symbol interpreted as the multiplication of integers.

Show that $\operatorname{Th}(\mathbb{Z}, +, \cdot)$ is undecidable. *Hint:* Apply Lagrange's four-square theorem:

Theorem 1 (Lagrange's four-square theorem)

Every natural number can be represented as the sum of four integer squares, that is, for every $x \in \mathbb{N}$, there are integers $x_1, x_2, x_3, x_4 \in \mathbb{Z}$, such that $x = x_1^2 + x_2^2 + x_3^2 + x_4^2$.

Task 2

Consider the structure $(\mathbb{N}, +, \cdot, s, 0)$. Use Gödel's β -function in order to formalize the following statements in predicate logic:

- (a) $x^y = z$ (use free variables x, y and z),
- (b) Fermat's Last Theorem,
- (c) Collatz conjecture.