Universität Siegen
Lehrstuhl Theoretische Informatik
Markus Lohrey

Advanced Logic
SS 2023

Exercise 7

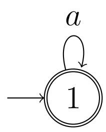
Task 1
Which of the following statements are correct? Give reasons for your answer.

(a) $(\mathbb{N}, \leq)$ is automatically presentable.

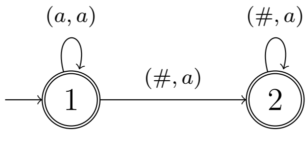
(b) Let $M \subseteq \mathbb{N}$ (unary relation), then $(\mathbb{N}, M)$ is automatically presentable.

Solution:

(a) This statement is correct: Let $f \colon \mathbb{N} \to \{a\}^*$ be defined by $f(i) = a^i$. Let $(\{a\}^*, \leq_a)$ with $a^i \leq_a a^j$ if and only if $i \leq j$. Then $(\mathbb{N}, \leq)$ and $(\{a\}^*, \leq_a)$ are isomorphic and f is the corresponding isomorphism, as $i \leq j$ if and only if $f(i) = a^i \leq a^j = f(j)$. Furthermore, f is bijective. Moreover, $(\{a\}^*, \leq_a)$ is automatic, as



is a finite automaton for $\{a\}^*$ and



is a 2-tape automaton for $\leq_a$.

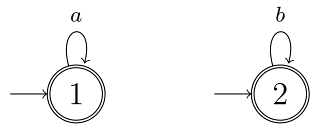
(b) This statement is correct: If M or $\mathbb{N} \setminus M$ is finite, then let $(\{a\}^*, P)$ with $P = \{a^i \mid i \in M\}$ and $f(i) = a^i$. We find that $(\{a\}^*, P)$ and $(\mathbb{N}, M)$ are isomorphic and f is the corresponding isomorphism. The automaton that accepts $\{a\}^*$ is shown in part (a). If M is finite, then P is finite and thus accepted by a finite automaton as finite languages are always regular (recall that P is a unary relation and a 1-tape automaton is a „standard“ finite automaton).

If $\mathbb{N} \setminus M$ is finite, then the complement of P is finite and hence regular. As regular languages are closed under taking the complement, we find that P is regular and thus there is a finite automaton which accepts P. Thus, $(\{a\}^*, P)$ is automatic in this case.

If both M and $\mathbb{N} \setminus M$ are infinite, then let $M = \{a_0, a_1, a_2, \dots\}$ and let $\mathbb{N} \setminus M = \{b_1, b_2, \dots\}$ (note that both M and $\mathbb{N} \setminus M$ are countable as subsets of $\mathbb{N}$). We define $(\{a\}^* \cup \{b\}^*, P)$ by $P = \{a\}^*$ and $f \colon \mathbb{N} \to \{a\}^* \cup \{b\}^*$ by

$$f(i) = \begin{cases} a^j & \text{if } i \in M, a_j = i, \\ b^j & \text{if } i \notin M, b_j = i. \end{cases}$$

Then f is an isomorphism, as f is bijective and $f(i) \in P$ holds if and only if $i \in M$. Furthermore, we find that $(\{a\}^* \cup \{b\}^*, P)$ is automatic, as



is an automaton for $\{a\}^* \cup \{b\}^*$ and $P = \{a\}^*$ is accepted by the finite automaton in part (a).

Task 2
Are any two countable linear orders without a smallest and a largest element isomorphic?

Solution:
We find that $(\mathbb{Z}, \leq)$ and $(\mathbb{Q}, \leq)$ are countable linear orders without a smallest and a largest element, but they are not isomorphic: For example, we find that $(\mathbb{Q}, \leq)$ is dense, but $(\mathbb{Z}, \leq)$ is not dense. In order to show a contradiction, assume that there is a bijection $h : \mathbb{Z} \to \mathbb{Q}$, such that

$$a \leq b \iff h(a) \leq h(b)$$

holds for all $a, b \in \mathbb{Z}$. Fix two elements $a, b \in \mathbb{Z}$ such that $a + 1 = b$. As $\mathbb{Q}$ is dense, there is an element $q \in \mathbb{Q}$, such that $h(a) < q < h(b)$. As h is a bijection, we have $q = h(c)$ for an element $c \in \mathbb{Z}$. However, we either have $c < a$ or $b < c$, as $a + 1 = b$. This yields a contradiction.

Task 3
Let $\Sigma = \{a, b\}$. Show that

(a) the *lexicographic order* $\leq_{\mathsf{lex}}$ defined by

$$\begin{aligned} u \leq_{\mathsf{lex}} v \iff & u \text{ is a prefix of } v \text{ or} \\ & \text{there are } x, y, z \in \Sigma^* \text{ such that } u = xay \text{ and } v = xbz, \end{aligned}$$

(b) the *length-lexicographic order* $\leq_{\mathsf{llex}}$ defined by

$$u \leq_{\mathsf{llex}} v \iff |u| < |v| \text{ or } (|u| = |v| \text{ and } u \leq_{\mathsf{lex}} v).$$

are linear orders.

Solution:
We have to show that the orders are *reflexive*, *anti-symmetric*, *transitive* and *linear*.

(a) The lexicographic order is a linear order:

reflexive: We have $u \leq_{\mathsf{lex}} u$ for every $u \in \Sigma^*$, as u is a prefix of itself.

anti-symmetric: Let $u \leq_{\mathsf{lex}} v$ and $v \leq_{\mathsf{lex}} u$ for $u, v \in \Sigma^*$. Then u must be a prefix of v and v must be a prefix of u. It follows that $u = v$.

transitive: Let $u \leq_{\mathsf{lex}} v$ and $v \leq_{\mathsf{lex}} w$ for $u, v, w \in \Sigma^*$. Several cases are possible:

(i) u is a prefix of v and v is a prefix of w

(ii) u is a prefix of v and $v = xay$, $w = xbz$ for $x, y, z \in \Sigma^*$

(iii) $u = xay$, $v = xbz$ for $x, y, z \in \Sigma^*$ and v is a prefix of w

(iv) $u = xay$, $v = xbz$ for $x, y, z \in \Sigma^*$ and $v = paq$, $v = pbr$ for $p, q, r \in \Sigma^*$

In case (i), we find that u thus must be a prefix of w and hence $u \leq_{\mathsf{lex}} w$.
In case (ii), as u is a prefix of v, it follows that u is a prefix of xay. If u is even a prefix of x, it follows that u is a prefix of $w = xbz$ and hence $u \leq_{\mathsf{lex}} w$. Otherwise, u is of the form xay' for some $y' \in \Sigma^*$ and hence, $u \leq_{\mathsf{lex}} w$.
In case (iii), as v is a prefix of w, we find that $w = xbzz'$ for some $z' \in \Sigma^*$. In particular, we have $u = xay$ and $v = xbzz'$, so $u \leq_{\mathsf{lex}} w$.
In case (iv), as $v = xbz$ and $v = paq$, we either have that xb is a prefix of p or pa is a prefix of x. If pa is a prefix of x, we find that u is of the form $u = pasy$ for some $s \in \Sigma^*$. Hence, $u \leq_{\mathsf{lex}} w$. The other case that xb is a prefix of p is symmetric.

linear: For all $u, v \in \Sigma^*$ it holds that $u \leq_{\mathsf{lex}} v$ or $v \leq_{\mathsf{lex}} u$.

(b) The length-lexicographic order is a linear order:

reflexive: We have $|u| = |u|$ and $u \leq_{\mathsf{lex}} u$ for every $u \in \Sigma^*$.

anti-symmetric: Let $u \leq_{\mathsf{llex}} v$ and $v \leq_{\mathsf{llex}} u$ for $u, v \in \Sigma^*$. Then it must hold that $|u| = |v|$ and $u \leq_{\mathsf{lex}} v$ and $v \leq_{\mathsf{lex}} u$. As in part (a), it follows that $u = v$.

transitive: Let $u \leq_{\mathsf{llex}} v$ and $v \leq_{\mathsf{llex}} w$ for $u, v, w \in \Sigma^*$. Again, several cases are possible:

(i) $|u| < |v|$ and $|v| < |w|$

(ii) $|u| = |v|$, $u \leq_{\mathsf{lex}} v$ and $|v| < |w|$

(iii) $|u| < |v|$ and $|v| = |w|$, $v \leq_{\mathsf{lex}} w$

(iv) $|u| = |v|$, $u \leq_{\mathsf{lex}} v$ and $|v| = |w|$, $v \leq_{\mathsf{lex}} w$.

In cases (i), (ii) and (iii), it follows immediately that $|u| < |w|$ and hence $u \leq_{\mathsf{llex}} v$. In case (iv), we have $u \leq_{\mathsf{lex}} v$ and $v \leq_{\mathsf{lex}} w$, so the statement follows from part (a).

linear: For all $u, v \in \Sigma^*$ it holds that $u \leq_{\mathsf{llex}} v$ or $v \leq_{\mathsf{llex}} u$.