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Exercise 10

Task 1
Let $\Sigma = \{a, b, c\}$. Find MSO-formulas corresponding the following regular languages:

(a) $L = \{w \in \Sigma^+ \mid \text{The first and last letter of } w \text{ are identical}\}$

(b) $L = \{a^n b^m c^\ell \mid n \geq 0, m \geq 1, \ell \geq 2\}$

(c) $L = \{w \in \Sigma^+ \mid w \text{ does not contain the word } bab\}$

(d) $L = \{w \in \Sigma^+ \mid w \text{ contains at most two distinct characters}\}$

Solution:

(a) $\exists x \exists y\, (\forall u (u \leq x) \wedge \forall z (z \leq y) \wedge ((P_a(x) \wedge P_a(y)) \vee (P_b(x) \wedge P_b(y)) \vee (P_c(x) \wedge P_c(y)))$

(b)

$$\forall x((P_a(x) \rightarrow (\forall u((u \leq x) \rightarrow P_a(u)))) \wedge (P_b(x) \rightarrow (\forall y((y \leq x) \rightarrow (P_a(y) \vee P_b(y))))))$$
$$\wedge\, \exists z P_b(z) \wedge \exists v \exists w((w \neq v) \wedge P_c(w) \wedge P_c(v))$$

(c) $\neg \exists x \exists y \exists z (y = x + 1 \wedge z = y + 1 \wedge P_b(x) \wedge P_a(y) \wedge P_b(z))$

(d) $\forall x\, (P_a(x) \vee P_b(x)) \vee \forall y (P_a(y) \vee P_c(y)) \vee \forall z\, (P_b(z) \vee P_c(z))$

Task 2
Which regular languages over $\Sigma = \{a, b, c\}$ correspond to the following MSO formulas?

(a) $\forall x \forall y (P_a(x) \wedge P_b(y) \wedge (x < y) \wedge (\forall z (x < z < y) \rightarrow \neg P_b(z)))$
$\rightarrow (\exists x_1 \exists x_2 (x < x_1 < x_2 < y) \wedge P_c(x_1) \wedge P_c(x_2))$

(b) $\exists X (\exists x \exists y (\forall u (x \leq u \leq y) \wedge x \in X \wedge y \in X) \wedge$
$\forall x \forall y (y = x + 1 \rightarrow (x \in X \leftrightarrow \neg(y \in X))))$

Solution:

(a) A word w from the regular language corresponding to the MSO-formula satisfies: There are at least two occurencences of the character c between an occurrence of a and the subsequent occurrence of b in w.

(b) The regular language contains all words of odd length.

Task 3

A strategy to find a MSO-formula for a given regular language is given in the proof of Büchi's Theorem. Use this strategy to find a MSO-formula for the language

$$L = \{w \in \{a,b,c\}^+ \mid \text{The number of } a\text{'s in } w \text{ is odd}\}.$$

Solution:

We use the strategy from Büchi's Theorem (slide 152):
The automaton $(\{1,2\}, \{a,b,c\}, \delta, 1, \{2\})$ with

$$\begin{aligned}
\delta(1,a) &= 2\\
\delta(1,b) &= 1\\
\delta(1,c) &= 1\\
\delta(2,a) &= 1\\
\delta(2,b) &= 2\\
\delta(2,c) &= 2
\end{aligned}$$

accepts the language L. We obtain the following formula:

$$\begin{aligned}
&\exists X_1 \exists X_2 (X_1 \cap X_2 = \emptyset \wedge \forall x (x \in X_1 \vee x \in X_2)\\
&\wedge \exists x (\forall y (x \leq y) \wedge ((P_a(x) \wedge x \in X_2) \vee (P_b(x) \wedge x \in X_1) \vee (P_c(x) \wedge x \in X_1)))\\
&\wedge \exists x (\forall y (y \leq x) \wedge x \in X_2)\\
&\wedge \forall x \forall y (y = x+1 \rightarrow (x \in X_1 \wedge P_a(y) \wedge y \in X_2)\\
&\qquad \vee (x \in X_1 \wedge P_b(y) \wedge y \in X_1)\\
&\qquad \vee (x \in X_1 \wedge P_c(y) \wedge y \in X_1)\\
&\qquad \vee (x \in X_2 \wedge P_a(y) \wedge y \in X_1)\\
&\qquad \vee (x \in X_2 \wedge P_b(y) \wedge y \in X_2)\\
&\qquad \vee (x \in X_2 \wedge P_c(y) \wedge y \in X_2)))
\end{aligned}$$

Task 4

The following MSO-formula was obtained via the technique from Büchi's Theorem (slide 152) from a regular language L. Which language is characterized by this formula?

$$\begin{aligned}
&\exists X_1 \exists X_2 \exists X_3 (X_1 \cap X_2 = \emptyset \wedge X_1 \cap X_3 = \emptyset \wedge X_2 \cap X_3 = \emptyset \wedge \forall x (x \in X_1 \vee x \in X_2 \vee x \in X_3)\\
&\wedge \exists x (\forall y (x \leq y) \wedge ((P_a(x) \wedge x \in X_2) \vee (P_b(x) \wedge x \in X_2)))\\
&\wedge \exists x (\forall y (y \leq x) \wedge (x \in X_2))\\
&\wedge \forall x \forall y (y = x+1 \rightarrow (x \in X_1 \wedge P_a(y) \wedge y \in X_2)\\
&\qquad \vee (x \in X_1 \wedge P_b(y) \wedge y \in X_2)\\
&\qquad \vee (x \in X_2 \wedge P_a(y) \wedge y \in X_3)\\
&\qquad \vee (x \in X_2 \wedge P_b(y) \wedge y \in X_3)\\
&\qquad \vee (x \in X_3 \wedge P_a(y) \wedge y \in X_3)\\
&\qquad \vee (x \in X_3 \wedge P_a(y) \wedge y \in X_3)))
\end{aligned}$$

Solution:
We find that the formula corresponds to the finite automaton $(\{1, 2, 3\}, \{a, b\}, \delta, 1, \{2\})$ with

$$\begin{aligned}
\delta(1, a) &= 2 \\
\delta(1, b) &= 2 \\
\delta(2, a) &= 3 \\
\delta(2, b) &= 3 \\
\delta(3, a) &= 3 \\
\delta(3, b) &= 3.
\end{aligned}$$

This automaton accepts the language $L = \{w \in \{a, b\}^+ \mid |w| \leq 1\}$.