Exercise 11

Task 1

Let $\Sigma = \{a, b, c\}$. Find nondeterministic Büchi automata that accept the following ω-languages:

(a) $L_a = \{w \in \Sigma^\omega \mid w \text{ does not contain the word } bab\}$

(b) $L_b = \{w \in \Sigma^\omega \mid w \text{ contains at most two distinct characters}\}$

(c) $L_c = \{w \in \Sigma^\omega \mid \text{ every } c \text{ in } w \text{ is immediately followed by the character } b\}$

(d) $L_d = \{w \in \Sigma^\omega \mid \text{ there are at least two } c\text{'s between an } a \text{ and the subsequent } b \text{ in } w\}$

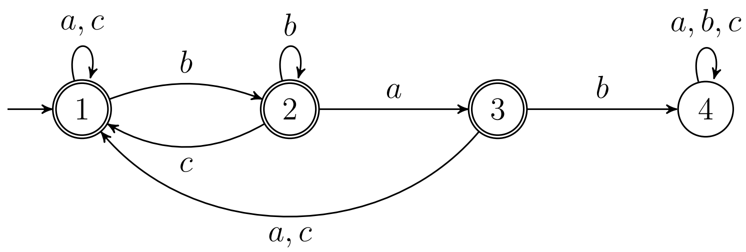
(e) $L_e = \{w \in \Sigma^\omega \mid w \text{ contains the words } aa \text{ or } bb \text{ infinitely often}\}$

(f) $L_f = \{w \in \Sigma^\omega \mid w \text{ contains } a \text{ infinitely often if and only if } w \text{ contains } b \text{ infinitely often}\}$

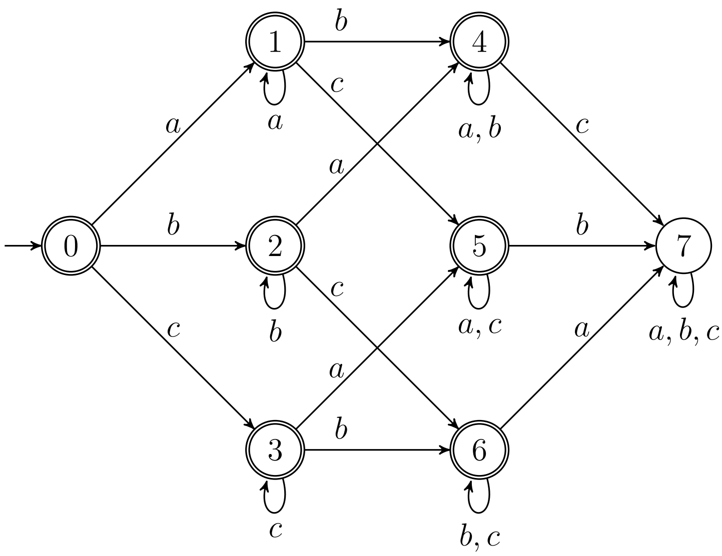
(g) $L_g = \{w \in \Sigma^\omega \mid w \text{ contains the word } aba \text{ only finitely often}\}$

Solution:

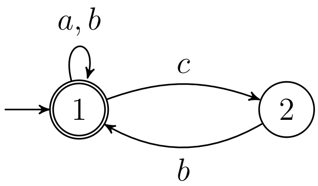
(a) Büchi automaton for L_a:



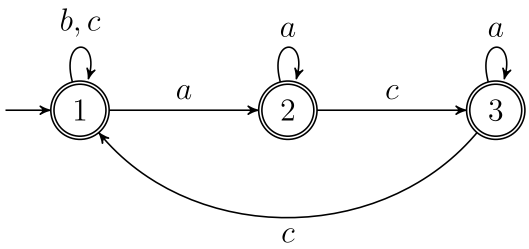
(b) Büchi automaton for L_b:



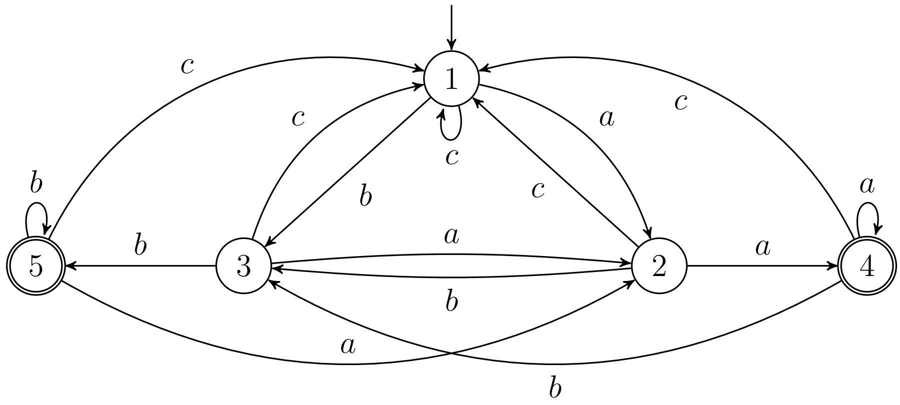
(c) Büchi automaton for L_c:



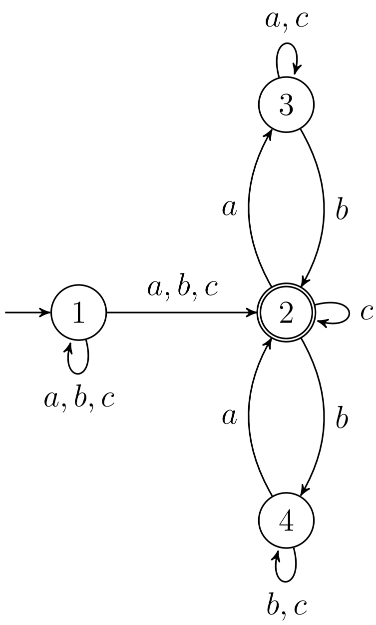
(d) Büchi automaton for L_d:



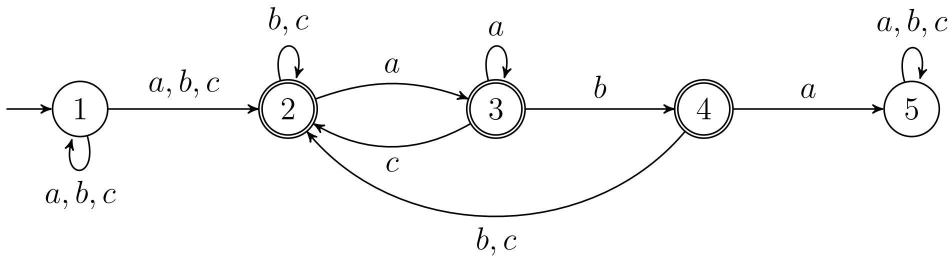
(e) Büchi automaton for L_e:



(f) Büchi automaton for L_f:



(g) Büchi automaton for L_g:



Task 2
Formulate the following properties of the structure $\mathcal{A}$ in ∃SO:

(a) Perfect Matching: The directed graph $\mathcal{A} = (V, E)$ has a perfect matching, i.e., there is a subset $M \subseteq E$ such that every node is an end point of exactly one edge from M.

(b) Hamilton Path: The directed graph $\mathcal{A} = (V, E)$ has a Hamilton path, i.e., there is an enumeration $v_1, \dots, v_n$ of V such that every node from V appears exactly once in $v_1, \dots, v_n$ and $(v_i, v_{i+1}) \in E$ for every $1 \leq i \leq n - 1$

(c) Graph Isomorphism: The structure $\mathcal{A} = (V, E, F)$ with E and F binary relations is such that the directed graphs (V, E) and (V, F) are isomorphic

(d) Subgraph Isomorphism: The structure $\mathcal{A} = (V, E, F, U)$ with E and F as above and $U \subseteq V$ a unary relation is such that the graph $(U, F \cap (U \times U))$ (the subgraph of (V, F) induced by U) is isomorphic to some subgraph of (V, E).

Solution:(a) The following $\exists$SO-sentence states that M is a subset of E, every vertex x is an endpoint of at least one edge in M and if there is an edge from x to y (or y to x) in M, then there is no edge from x to any other vertex $z \neq y$.

$$\begin{aligned}\exists M : \forall x, y : M(x, y) \rightarrow E(x, y) \wedge \\ \forall x \exists y : (M(x, y) \vee M(y, x)) \wedge \\ \forall x, y : (M(x, y) \vee M(y, x)) \rightarrow \neg \exists z((z \neq y) \wedge (M(x, z) \vee M(z, x)))\end{aligned}$$

(b) The following $\exists$SO-sentence states that there exists a linear order on the vertex set such that whenever v is the direct successor of u then $(u, v) \in E$.

$$\begin{aligned}\exists \leq : \forall x : x \leq x \wedge \\ \forall x, y, z : (x \leq y \wedge y \leq z) \rightarrow x \leq z \wedge \\ \forall x, y : (x \leq y \wedge y \leq x) \rightarrow x = y \wedge \\ \forall x, y : (x \leq y \vee y \leq x) \wedge \\ \forall x, y : (x \leq y \wedge x \neq y \wedge \neg \exists z : z \neq x \wedge z \neq y \wedge x \leq z \wedge z \leq y) \rightarrow E(x, y)\end{aligned}$$

(c) The two graphs are isomorphic, if there is a bijective mapping $\varphi \colon V \rightarrow V$ such that $(u, v) \in E$ if and only if $(\varphi(u), \varphi(v)) \in F$. The mapping φ can be identified with a binary relation $S \subseteq V \times V$, such that $S(x, y)$ means $y = \varphi(x)$.

$$\begin{aligned}\exists S : \forall x \exists y S(x, y) \wedge \\ \forall x, y : S(x, y) \rightarrow \neg \exists z : (z \neq y \wedge S(x, z)) \wedge \\ \forall x \exists y : S(y, x) \wedge \\ \forall x, y, z : (S(y, x) \wedge S(z, x)) \rightarrow y = z \wedge \\ \forall x, y, z, v : (S(x, z) \wedge S(y, v)) \rightarrow (E(x, y) \leftrightarrow F(z, v))\end{aligned}$$

The first two lines in the above formula state that S corresponds to a mapping (every $x \in V$ is mapped to exactly one value $\varphi(x) \in V$). The next two lines state that the mapping is bijective. The last line of the formula states that if x is mapped to z and y is mapped to v via the bijective mapping, then there is an edge from x to y if and only if there is an edge from z to u (in the corresponding edge sets E and F).

(d) If $(U, F \cap (U \times U))$ is isomorphic to a subgraph of (V, E), then there is a subset $W \subseteq V$ and a bijective mapping $\varphi\colon U \to W$, such that for every two nodes $x, y \in U$ we have $(x, y) \in F$ if and only if $(\varphi(x), \varphi(y)) \in E$. The bijective mapping φ is again represented by a binary relation S.

$$\begin{aligned}\exists W \exists S : {} & \forall x : (U(x) \to \exists y : (W(y) \land S(x, y))) \land \\ & \forall x, y : (U(x) \land W(y) \land S(x, y)) \to \neg\exists z : (z \neq y \land S(x, z)) \land \\ & \forall x : (W(x) \to \exists y : (U(y) \land S(y, x))) \land \\ & \forall x, y, z : (W(x) \land U(y) \land U(z) \land S(y, x) \land S(z, x)) \to y = z \land \\ & \forall x, y, z, v : (U(x) \land U(y) \land W(z) \land W(v) \land S(x, z) \land S(y, v)) \to (F(x, y) \leftrightarrow E(z, v))\end{aligned}$$