

## Exercise 5

### Task 1

Let  $G = (V, E)$  be an undirected graph with

$$V = \{1, 2, 3, 4, 5, 6\}, \quad E = \{\{1, 3\}, \{1, 6\}, \{2, 3\}, \{2, 5\}, \{3, 5\}, \{4, 6\}, \{5, 6\}\}.$$

- (a) Compute the Tutte matrix  $T_G$  of  $G$ .
- (b) Compute the polynomial  $\det(T_G)$ .
- (c) Does  $G$  have a perfect matching? If yes, name all perfect matchings of  $G$ . If no, justify your answer.

### Task 2

Let  $G = (V, E)$  be an undirected graph with  $V = \{1, \dots, n\}$ . Let  $T^G = (T_{u,v})_{1 \leq u, v \leq n}$  be the matrix defined by

$$T_{u,v} = \begin{cases} x_{u,v} & \text{if } \{u, v\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

- (a) Let  $G$  be a bipartite graph. This means, there are disjoint subsets  $U, W \subset V$  such that  $V = U \cup W$  and  $\{u, w\} \in E$  only if  $u \in U$  and  $w \in W$  (or  $w \in U$  and  $u \in W$ ). Show that  $G$  has a perfect matching if and only if  $\det(T^G) \neq 0$ .
- (b) Does (a) also hold, if  $G$  is not bipartite?