Exercise 6

Task 1
Let $f : \{0,1\}^* \to \mathbb{Z}^{2\times 2}$ be the homomorphism defined by

$$f(0) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad f(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Show that the entries of the matrix $f(w)$ are upper bounded by the $(|w|+1)$-th Fibonacci number $F_{|w|+1}$. Furthermore, give an example for a string w, where at least one entry of $f(w)$ takes indeed the value $F_{|w|+1}$.

Solution
Note that here we define $F_0 = 0$, $F_1 = 1$ and $F_{i+1} = F_i + F_{i-1}$ (there is another convention for the starting values!).
Part 1 of the task is an induction on $|w| = n$. We will actually prove a stronger claim: The entries of $f(w) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are bounded by

- either $a \le F_{n+1}, b, c \le F_n, d \le F_{n-1}$
- or $b \le F_{n+1}, a, d \le F_n, c \le F_{n-1}$
- or $c \le F_{n+1}, a, d \le F_n, b \le F_{n-1}$
- or $d \le F_{n+1}, b, c \le F_n, a \le F_{n-1}$.

By looking at the identity matrix, it is clear that the assumption is true for $n = 0$, if we set $F_{-1} = 1$. By the inductions hypotheses, we can assume that

$$f(w) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \le \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

or one of the other 3 cases hold. Hence, for the induction step we will consider another 2 cases each:

$$f(0w) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ a+c & b+d \end{pmatrix} \le \begin{pmatrix} F_{n+1} & F_n \\ F_{n+2} & F_{n+1} \end{pmatrix}$$

and

$$f(1w) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix} \le \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_n & F_{n-1} \end{pmatrix},$$

where both matrices satisfy one of the four stated conditions. The other 6 cases work analogously.
Part 2: Take for instance $w = (10)^n$ for even $|w|$ and $w = 0(10)^n$ for odd $|w|$. This yields

$$f((10)^n) = \begin{pmatrix} F_{2n+1} & F_{2n} \\ F_{2n} & F_{2n-1} \end{pmatrix}, \qquad f(0(10)^n) = \begin{pmatrix} F_{2n+1} & F_{2n} \\ F_{2n+2} & F_{2n+1} \end{pmatrix}.$$