Exercise 2

Task 1. If $|x\rangle$ is a unit vector and $\{|x_1\rangle, ..., |x_d\rangle\}$ an orthonormal base then $|x\rangle = \Sigma_{i=1}^d \alpha_i |x_i\rangle$ for unique $\alpha_1, ..., \alpha_d \in \mathbb{C}$ with $\Sigma_{i=1}^d |\alpha_i|^2 = 1$.

Lösung. We can express $|x\rangle$ in terms of the orthonormal base $\{|x_1\rangle, ..., |x_d\rangle\}$, then

$$|x\rangle = \Sigma_i^d \alpha_i |x_i\rangle$$

Let's take the inner product of both sides with each basis vector $|x_j\rangle$ for $j = 1, ..., d$.

$$\begin{aligned}\langle x_j|x\rangle &= \langle x_j| \left(\Sigma_i^d \alpha_i |x_i\rangle\right) \\ &= \Sigma_i^d \alpha_i \langle x_j|x_i\rangle\end{aligned}$$

As the basis is orthonormal, then $\langle x_j|x_i\rangle = 0$ for $i \neq j$ and

$$\begin{aligned}\langle x_j|x\rangle &= \alpha_j \\ \implies |\alpha_j|^2 &= |\langle x_j|x\rangle|^2\end{aligned}$$

Remember that $|\langle x_j|x\rangle|^2$ represents the probability of measuring $|x\rangle$ in the state $|x_j\rangle$. As $|x\rangle$ is the unit vector, then the sum of these probabilities over all the base states must be 1.

$$\implies \Sigma_j^d |\alpha_j|^2 = 1$$

Task 2. Let f be a linear mapping and let $\{|x_1\rangle, .., |x_d\rangle\}$ and $\{|y_1\rangle, ... |y_d\rangle\}$ be two bases of $\mathbb{C}^d$. Let A (resp., B) be the matrix for f in the basis $\{|x_1\rangle, .., |x_d\rangle\}$ (resp., $\{|y_1\rangle, ... |y_d\rangle\}$).
Then, there is an invertible matrix $C \in \mathbb{C}^{d\times d}$ such that $B = C^{-1}AC$.
Find the matrix C explicitly.

Lösung. Each vector $|y_i\rangle$ can be expressed as a linear combination of the $|x_j\rangle$ basis vectors:

$$|y_i\rangle = \sum_{j=1}^{d} c_{ji} |x_j\rangle$$

where c_{ji} are the components of the vector $|y_i\rangle$ in the $|x_j\rangle$ basis.

The matrix C is constructed such that its columns are the coordinates of the $|x_i\rangle$ vectors expressed in the $\{|y_j\rangle\}$ basis:

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1d} \\ c_{21} & c_{22} & \cdots & c_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ c_{d1} & c_{d2} & \cdots & c_{dd} \end{bmatrix}$$

where the i-th column of C is made of the coefficients $[c_{1i}, c_{2i}, \ldots, c_{di}]^T$, which are the coordinates of $|x_i\rangle$ in the $\{|y_j\rangle\}$ basis. The coefficients c_{ij} can be obtained by taking inner products between the vectors $|y_i\rangle$ and $|x_j\rangle$:

$$c_{ij} = \langle y_i| \, |x_j\rangle$$

$$C = \begin{pmatrix} \langle y_1||x_1\rangle & \langle y_1||x_2\rangle & \cdots & \langle y_1||x_d\rangle \\ \langle y_2||x_1\rangle & \langle y_2||x_2\rangle & \cdots & \langle y_2||x_d\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle y_d||x_1\rangle & \langle y_d||x_2\rangle & \cdots & \langle y_d||x_d\rangle \end{pmatrix}$$

Task 3. The trace of a matrix $\text{tr}(A)$ is defined as the sum of the diagonal entries:

$$\text{tr}(A) = \Sigma_{i=1}^{d} A_{i,i},$$

then, prove the following important properties:

- $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$
- $\text{tr}(\alpha A) = \alpha \cdot \text{tr}(A)$
- $\text{tr}(AB) = \text{tr}(BA)$

Lösung.

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$$\begin{aligned} \text{tr}(A + B) &= \text{tr}(A_{ij} + B_{ij}) \text{ for } i, j = 1, .., n \\ &= (a_{11} + b_{11}) + (a_{22} + b_{22}) + ... + (a_{nn} + b_{nn}) \\ &= (a_{11} + ... + a_{nn}) + (b_{11} + ... + b_{nn}) \\ &= \text{tr}(A) + \text{tr}(B) \end{aligned}$$

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$$\begin{aligned} \text{tr}(\alpha A) &= \text{tr}(\alpha A_{i,j}) \text{ for } i, j = 1, ..., d \\ &= \Sigma_i^d(\alpha A_{i,i}) = \alpha \Sigma_i^d(A_{i,i}) \\ &= \alpha \text{tr}(A) \end{aligned}$$

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\begin{aligned}
\operatorname{tr}(AB) =& \operatorname{tr}\left(\Sigma_k^d A_{i,k} B_{k,j}\right) \text{ for } i,j = 1, ..., d \\
=&(a_{1,1}b_{1,1} + ... + a_{1,n}b_{n,1}) + (a_{2,1}b_{1,2} + ... + a_{2,n}b_{n,2}) \\
&+ ... + (a_{n,1}b_{1,n} + ... + a_{n,n}b_{n,n}) \\
=&(b_{1,1}a_{1,1} + ... + b_{n,1}a_{1,n}) + (b_{1,2}a_{2,1} + ... + b_{n,2}a_{2,n}) \\
&+ ... + (b_{1,n}a_{n,1} + ... + b_{n,n}a_{n,n}) \\
=&(b_{1,1}a_{1,1} + b_{1,2}a_{2,1} + ... + b_{1,n}a_{n,1} \\
&+ ... + (b_{n,1}a_{1,n} + b_{n,2}a_{2,n} + ... + b_{nn}a_{nn)} \\
=&\operatorname{tr}\left(\Sigma_k^d B_{i,k} A_{k,j}\right) \text{ for } i,j = 1, ..., d \\
=&\operatorname{tr}(BA)
\end{aligned}
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Task 4. If Π is any projector ($\Pi^2 = \Pi$) find a subspace S with $\Pi = \Pi_S$.

Lösung. We can define the subspace S for which Π acts as following:

$$S = \operatorname{Im}(\Pi)$$

Here, $\operatorname{Im}(\Pi)$ represents the image of Π, consisting of all vectors $|x\rangle$ in $\mathbb{C}^d$ such that $|x\rangle = \Pi(|y\rangle)$ for some $|y\rangle \in \mathbb{C}^d$. This means every vector in S remains unchanged when it is projected by Π, i.e., $\Pi(|x\rangle) = |x\rangle$ for all $|x\rangle \in S$. Now we can define the projector Π_S onto the subspace S by:

$$\Pi_S = \sum_{i=1}^{k} |x_i\rangle \langle x_i|$$

where $\{|x_1\rangle, \ldots, |x_k\rangle\}$ is an orthonormal basis for S. This basis can be obtained from the image of Π (for example, by applying a method like the Gram-Schmidt process to the set of vectors $\Pi(e_j)$ for $j = 1, \ldots, d$, where $\{e_j\}$ is the standard basis of $\mathbb{C}^d$). Since Π is a projector, it satisfies $\Pi^2 = \Pi$. For the subspace S we defined as $\operatorname{Im}(\Pi)$, any vector $v \in S$ is such that $\Pi(v) = v$. Thus, Π_S constructed from a basis of S will also satisfy this property. For any vector $|w\rangle \in \mathbb{C}^d$, $\Pi(|w\rangle)$ lies in S. Hence, $\Pi_S(\Pi(|w\rangle)) = \Pi(|w\rangle)$. Since Π and Π_S both act identically on vectors in S and effectively zero out components orthogonal to S, they must be the same operator.

Task 5. Calculate the eigenvalues of the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Lösung. We find the eigenvalues of a matrix A (when they exist) by solving the equation $\det A - \lambda I = 0$ for λ:

- For σ_x we have

$$\begin{aligned} \det(\sigma_x - \lambda I) &= \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \\ &\implies \lambda^2 - 1 = 0 \\ &\implies \lambda = 1 \text{ or } \lambda = -1 \end{aligned}$$

- For σ_y we have

$$\begin{aligned} \det(\sigma_y - \lambda I) &= \det \begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix} \\ &\implies \lambda^2 - 1 = 0 \\ &\implies \lambda = 1 \text{ or } \lambda = -1 \end{aligned}$$

- For σ_z we have

$$\begin{aligned} \det(\sigma_z - \lambda I) &= \det \begin{pmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} \\ &\implies (1-\lambda)(-1-\lambda) = 0 \\ &\implies \lambda = 1 \text{ or } \lambda = -1 \end{aligned}$$

Task 6. Prove that for every matrix $A \in \mathbb{C}^{d\times d}$ the matrix $A^\dagger A$ is positive semi-definite.

Lösung. We prove that $A^\dagger A$ is Hermitian and $\langle x| A^\dagger A |x\rangle \geq 0$ for every $|x\rangle$

- $(A^\dagger A)^\dagger = A^\dagger (A^\dagger)^\dagger$
  We used the property that $(AB)^\dagger = B^\dagger A^\dagger$ and $(A^\dagger)^\dagger = A$. This proves that $A^\dagger A$ is Hermitian.

- For any vector $|x\rangle$ in $\mathbb{C}^d$, consider the quadratic form $\langle x| A^\dagger A |x\rangle$:

  $$\langle x| A^\dagger A |x\rangle = ((\langle x| A^\dagger)(A |x\rangle)$$

  We can rewrite as:

  $$\langle x| A^\dagger A |x\rangle = \| A |x\rangle \|^2$$

  Here, $\| A |x\rangle \|^2$ denotes the squared norm of the vector $A |x\rangle$, which is a non-negative real number. The squared norm is calculated as:

  $$\| A |x\rangle \|^2 = ((\langle x| A^\dagger)(A |x\rangle)$$

  Since $\|(Ax)\|^2$ is always non-negative, and since $A^\dagger A$ is Hermitian, $A^\dagger A$ is positive semi-definite.

Task 7. Let A and B be unitary Hermitian positive definite projectors, then $A \otimes B$ is an unitary Hermitian positive definite projector.

Lösung. 1.

- Unitarity of $A \otimes B$:

  A matrix U is unitary if $U^\dagger U = I$, where $U^\dagger$ denotes the conjugate transpose of U and I is the identity matrix. Since A and B are unitary, we have $A^\dagger A = I$ and $B^\dagger B = I$. Now:

  $$(A \otimes B)^\dagger (A \otimes B) = (A^\dagger \otimes B^\dagger)(A \otimes B)$$

  Using the properties of the conjugate transpose and the tensor product, we have:

  $$(A^\dagger \otimes B^\dagger)(A \otimes B) = (A^\dagger A) \otimes (B^\dagger B) = I \otimes I = I$$

  Therefore, $(A \otimes B)^\dagger (A \otimes B) = I$.

- Hermitian property of $A \otimes B$:

  A matrix M is Hermitian if $M^\dagger = M$, where $M^\dagger$ denotes the conjugate transpose of M. Since A and B are Hermitian, we have $A^\dagger = A$ and $B^\dagger = B$. Now, consider the conjugate transpose of $A \otimes B$:

  $$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B$$

  Therefore, $(A \otimes B)^\dagger = A \otimes B$, proving that $A \otimes B$ is Hermitian.

- Positive definiteness of $A \otimes B$:

  A matrix M is positive definite if it is Hermitian and all its eigenvalues are strictly positive. Since A and B are Hermitian, their tensor product $A \otimes B$ is also Hermitian. Now, let's consider the eigenvalues of $A \otimes B$.

  The eigenvalues of $A \otimes B$ are products of the eigenvalues of A and B. Since A and B are positive definite projectors, their eigenvalues are all either 0 or 1. Therefore, the eigenvalues of $A \otimes B$ are also either 0 or 1, making $A \otimes B$ positive definite.

- $A \otimes B$ is a projector:

  A matrix P is called a projector if it satisfies $P^2 = P$. Since A and B are projectors, they satisfy $A^2 = A$ and $B^2 = B$. Then:

  $$(A \otimes B)^2 = (A \otimes B)(A \otimes B)$$

  Using the properties of the tensor product, we have:

  $$(A \otimes B)(A \otimes B) = AA \otimes BB = A^2 \otimes B^2 = A \otimes B$$

  Therefore, $(A \otimes B)^2 = A \otimes B$, proving that $A \otimes B$ is a projector.