Exercise 3

Task 1. Determine which of the following states are entangled and which are unentangled.

- $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$
- $\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$
- $\frac{1}{\sqrt{2}}(|00\rangle - i\,|10\rangle)$

Task 2. (Slide 60)
Show that: $|x\rangle = e^{i\phi}\,|y\rangle$ for some $\phi \iff |x\rangle\langle x| = |y\rangle\langle y|$

Task 3. There exists a geometrical representation where a unit vector inside a bounding sphere describes the quantum state of a single qubit (a pure state space of a two-level quantum mechanical system). Such a representation is called the Bloch Sphere, where north pole and and south pole correspond to the the pure states $|0\rangle$ and $|1\rangle$ respectively.

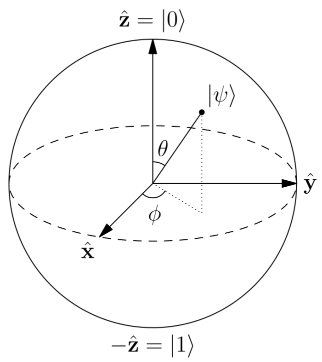


Abbildung 1: Bloch Sphere picture of a qubit.

Show that an arbitrary pure state of a single qubit $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ can be written as follows:

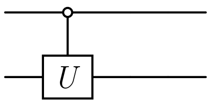
$$\psi = e^{i\gamma}\left(\cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle\right)$$

Task 4. Let be A and B unitary matrices corresponding to arbitrary $1-$qubit quantum gates. Then, we can define a quantum gate by using their direct sum:

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ 0 & 0 & B_{11} & B_{12} \\ 0 & 0 & B_{21} & B_{22} \end{pmatrix}$$

Show that the controlled operation applied by this gate is as follows: The operation A is applied to the second qubit if the first qubit is in state $|0\rangle$. Conversely, if the first qubit is in the state $|1\rangle$ then it applies the operation B to the second qubit. Draw the quantum circuit which represents this gate.

Hint: We can draw a gate that acts on a second qubit if the first qubit is in the state $|0\rangle$ using an *empty vertex* in the target qubit (See slide 73 from the lecture), for example:



Task 5. Prove that the spectral norm of a matrix $A \in \mathbb{C}^{d \times d}$ is preserved under similarity transformations.