Exercise 1

Task 1
Find a model for each of the following formulas of predicate logic, and structures in which the formulas evaluate to false.

(a) $\exists x \forall y (f(f(y)) = x)$

(b) $\exists x \exists y (P(x, y) \land \neg P(y, x))$

(c) $\forall x \big(f(g(f(x))) \neq g(f(g(x))) \big)$

(d) $R(x) \land Q(y) \land \forall x (\neg R(x) \lor \neg Q(x))$

Solution:

(a) Model: $\mathcal{A} = (\mathbb{N}, I_{\mathcal{A}})$, with $f^{\mathcal{A}}(x) = 1$ (for every $x \in \mathbb{N}$). The formula evaluates to true in the structure $\mathcal{A}$: There exists an element $x \in \mathbb{N}$ (which is $x = 1$), such that $f^{\mathcal{A}}(f^{\mathcal{A}}(y)) = x$ for every $y \in \mathbb{N}$. Another model: $\mathcal{A}' = (\{1\}, I_{\mathcal{A}'})$, with $f^{\mathcal{A}'}(1) = 1$.

A structure in which the formula evaluates to false: $\mathcal{B} = (\mathbb{N}, I_{\mathcal{B}})$, with $f^{\mathcal{B}}(x) = x$. The formula evaluates to false in the structure $\mathcal{B}$: There is no element $x \in \mathbb{N}$, such that for every $y \in \mathbb{N}$, we have $f^{\mathcal{B}}(f^{\mathcal{B}}(y)) = x$. Another example: $\mathcal{B}' = (\{0, 1\}, I_{\mathcal{B}'})$, with $f^{\mathcal{B}'}(x) = 1 - x$.

(b) Model: $\mathcal{A} = (\mathbb{N}, I_{\mathcal{A}})$, with $P^{\mathcal{A}} = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x < y\}$. The formula evaluates to true in the structure $\mathcal{A}$: There are elements $x, y \in \mathbb{N}$ (for example $x = 1, y = 2$) which satisfy $x < y$ (and thus, $(x, y) \in P^{\mathcal{A}}$), but which do not satisfy $y < x$ (such that $(y, x) \notin P^{\mathcal{A}}$). Another example: $\mathcal{A}' = (\{0, 1\}, I_{\mathcal{A}'})$, with $P^{\mathcal{A}'} = \{(1, 0), (0, 0)\}$.

A structure in which the formula evaluates to false: $\mathcal{B} = (\mathbb{N}, I_{\mathcal{B}})$, with $P^{\mathcal{B}} = \emptyset$. The formula evaluates to false in the structure $\mathcal{B}$, as the relation $P^{\mathcal{B}}$ is empty: Hence there is no pair (x, y) with $(x, y) \in P^{\mathcal{B}}$. Another example: $\mathcal{B}' = (\{1\}, I_{\mathcal{B}'})$, with $P^{\mathcal{B}'} = \{(1, 1)\}$.

(c) Model: $\mathcal{A} = (\mathbb{N}, I_{\mathcal{A}})$ with $f^{\mathcal{A}}(x) = 1$ and $g^{\mathcal{A}}(x) = 2$ for every $x \in \mathbb{N}$. The formula $\mathcal{A}$ evaluates to true in the structure, as $f^{\mathcal{A}}(g^{\mathcal{A}}(f^{\mathcal{A}}(x))) = 1$ for every $x \in \mathbb{N}$ and $g^{\mathcal{A}}(f^{\mathcal{A}}(g^{\mathcal{A}}(x))) = 2$ for every $x \in \mathbb{N}$. Another example: $\mathcal{A}' = (\mathbb{Z}, I_{\mathcal{A}'})$ with $f^{\mathcal{A}'}(x) = x - 1$ and $g^{\mathcal{A}'}(x) = x + 1$.

A structure in which the formula evaluates to false: $\mathcal{B} = (\mathbb{N}, I_\mathcal{B})$, with $f^\mathcal{B}(x) = x$ and $g^\mathcal{B}(x) = x$ for every $x \in \mathbb{N}$. The formula evaluates to false in this structure $\mathcal{B}$, as $f^\mathcal{B} = g^\mathcal{B}$ and hence $f^\mathcal{B}(g^\mathcal{B}(f^\mathcal{B}(x))) = g^\mathcal{B}(f^\mathcal{B}(g^\mathcal{B}(x)))$ for every $x \in \mathbb{N}$. Another example: $\mathcal{B}' = (\mathbb{R}, I_{\mathcal{B}'})$, wobei $f^{\mathcal{B}'}(x) = x^2$ und $g^{\mathcal{B}'}(x) = x^3$ (for $x = 1$, we have $f^\mathcal{B}(g^\mathcal{B}(f^\mathcal{B}(x))) = g^\mathcal{B}(f^\mathcal{B}(g^\mathcal{B}(x))))$.

(d) Model: $\mathcal{A} = (\mathbb{N}, I_\mathcal{A})$, with $x^\mathcal{A} = 2$, $y^\mathcal{A} = 3$, $R^\mathcal{A} = \{2x \mid x \in \mathbb{N}\}$ und $Q^\mathcal{A} = \mathbb{N} \setminus R^\mathcal{A}$. The formula evaluates to true in the structure: $R^\mathcal{A}$ is the set of even numbers, $Q^\mathcal{A}$ is the set of odd numbers. We find that $x^\mathcal{A} = 2$ is even and $y^\mathcal{A} = 3$ is odd. Furthermore, every $x \in \mathbb{N}$ is not even or not odd. Another model: $\mathcal{A}' = (\{0, 1\}, I_{\mathcal{A}'})$ with $x^{\mathcal{A}'} = 0$, $y^{\mathcal{A}'} = 1$, $R^{\mathcal{A}'} = \{0\}$ und $Q^{\mathcal{A}'} = \{1\}$.

A structure in which the formula evaluates to false: $\mathcal{B} = (\mathbb{N}, I_\mathcal{B})$, with $x^\mathcal{B} = y^\mathcal{B} = 1$ und $R^\mathcal{B} = Q^\mathcal{B} = \mathbb{N}$. The formula evaluates to false in the structure, as for every $x \in \mathbb{N}$, $R^\mathcal{B}(x)$ and $Q^\mathcal{B}(x)$ evaluate to true. Another example: $\mathcal{B}' = (\{0, 1\}, I_{\mathcal{B}'})$, with $x^{\mathcal{B}'} = y^{\mathcal{B}'} = 1$ and $R^{\mathcal{B}'} = Q^{\mathcal{B}'} = \{0\}$.

Task 2
Let f denote a binary function symbol and let R be a unary predicate symbol. Consider the following structures:

- $\mathcal{A}_1 = (\mathbb{N}, I_{\mathcal{A}_1})$, with $f^{\mathcal{A}_1}(x,y) = x \cdot y,\ R^{\mathcal{A}_1} = \{n \in \mathbb{N} \mid n \text{ is prime}\}$
- $\mathcal{A}_2 = (\mathbb{R}, I_{\mathcal{A}_2})$, with $f^{\mathcal{A}_2}(x,y) = x - 2y,\ R^{\mathcal{A}_2} = \{x \in \mathbb{R} \mid x \leq 0\}$

Do the following formulas evaluate to true in these structures?

(a) $\forall x(R(x) \vee R(f(x,x)))$

(b) $\forall x \exists y R(f(x,y))$

(c) $\forall x \forall y(f(x,y) = f(y,x))$

Solution:

(a) The structure $\mathcal{A}_1$ is not a model for this formula: The number $4 \in \mathbb{N}$ for example is not a prime and $f^{\mathcal{A}_1}(4,4) = 4 \cdot 4 = 16$ is not a prime either. The structure $\mathcal{A}_2$ is a model for this formula: We have $f^{\mathcal{A}_2}(x,x) = x - 2x = -x$, and for every real number x, we find that $x \leq 0$ or $-x \leq 0$, such that $x \in R^{\mathcal{A}_2}$ or $f^{\mathcal{A}_2}(x,x) \in R^{\mathcal{A}_2}$.

(b) The structure $\mathcal{A}_1$ is not a model for this formula: For example for $x = 4$ there is no $y \in \mathbb{N}$, such that $x \cdot y$ is a prime. The structure $\mathcal{A}_2$ is a model for this formula: For every $x \in \mathbb{R}$ there is $y \in \mathbb{R}$ such that $x - 2y \leq 0$.

(c) The structure $\mathcal{A}_1$ is a model for this formula, as multiplication of natural numbers is commutative, that is, $x \cdot y = y \cdot x$ for every $x, y \in \mathbb{N}$. The structure $\mathcal{A}_2$ is not a model for this formula: For example for $x = 1$ and $y = 2$, we find that $x - 2y = -3 \neq 0 = y - 2x$.

Task 3
Prove the theorem from Slide 4 (a language L is decidable if and only if L and its complement $\Sigma^* \setminus L$ are both recursively enumerable).

Solution:

- Assume that L is decidable, i.e, there is an algorithm A that stops for an input x after a finite number of steps with output YES if $x \in L$ and output NO if $x \notin L$.Then we easily obtain an algorithm A1 that terminates on input x if and only if $x \in L$ and an algorithm A2 that terminates on input x if and only if $x \notin L$. For A1 take the algorithm A and send it into a non terminatig loop (while true do nothing, end_while) if the original algorithm A wants to output NO (analogously for A2). This shows that L and its complement are recursively enumerable.

- Assume that L and its complement $\Sigma^* \setminus L$ are both recursively enumerable. Hence, there is an algorithm A1 (respectively, A2) that terminates on input x if and only if $x \in L$ (respectively $x \notin L$).

  Now consider the following algorithm:

  Input x
  $t \leftarrow 0$
  while True **do**
      Run $A1$ for t steps on input x;
      if $A1$ terminates on input x after t steps **then**
          Output YES
      end if
      Run $A2$ for t steps on input x
      if $A2$ terminates on input x after t steps **then**
          Output NO
      end if
      $t \leftarrow t + 1$;
  end while

  Since either $x \in L$ or $x \notin L$, either A1 terminates on input x (and then $x \in L$ holds) or A2 terminates on input x (and then $x \notin L$ holds). This shows that the above algorithm correctly decides whether $x \in L$ or $x \notin L$. Hence, L is decidable.