

Exercise 4

Task 1

Let \mathcal{A} be a structure. Show that $\text{Th}(\mathcal{A})$ is decidable if and only if $\text{Th}(\mathcal{A}_{rel})$ is decidable.

Task 2

Let $(\mathbb{N}, +, f)$ be a structure, where

- \mathbb{N} denotes the universe of the structure,
- $+$ denotes a binary function symbol interpreted as the addition of natural numbers, and
- f denotes a unary function symbol interpreted as the function $f: \mathbb{N} \rightarrow \mathbb{N}$ with $f(x) = x^2$.

Show that $\text{Th}(\mathbb{N}, +, f)$ is undecidable.

Task 3

Let $(\mathbb{Z}, +, \cdot)$ be a structure, where

- \mathbb{Z} denotes the universe of the structure,
- $+$ denotes a binary function symbol interpreted as the addition of integers, and
- \cdot denotes a binary function symbol interpreted as the multiplication of integers.

Show that $\text{Th}(\mathbb{Z}, +, \cdot)$ is undecidable.

Hint: Apply Lagrange's four-square theorem:

Theorem 1 (Lagrange's four-square theorem)

Every natural number can be represented as the sum of four integer squares, that is, for every $x \in \mathbb{N}$, there are integers $x_1, x_2, x_3, x_4 \in \mathbb{Z}$, such that $x = x_1^2 + x_2^2 + x_3^2 + x_4^2$.

Task 4

Consider the structure $(\mathbb{N}, +, \cdot, s, 0)$. Use Gödel's β -function in order to formalize the following statements in predicate logic:

- (a) $x^y = z$ (use free variables x, y and z),
- (b) Fermat's Last Theorem,
- (c) Collatz conjecture.