Exercise 4

Task 1
Let $\mathcal{A}$ be a structure. Show that $\text{Th}(\mathcal{A})$ is decidable if and only if $\text{Th}(\mathcal{A}_{rel})$ is decidable.

Solution:
We have to show two directions:

The direction „If $\text{Th}(\mathcal{A}_{rel})$ is decidable then $\text{Th}(\mathcal{A})$ is decidable" was already shown in the lecture (slides 32-33).

It remains to show the other direction: „If $\text{Th}(\mathcal{A})$ is decidable then $\text{Th}(\mathcal{A}_{rel})$ is decidable". For this, we transform a formula F (such that $\mathcal{A}_{rel}$ is suitable for F) into a formula F' (such that $\mathcal{A}$ is suitable for F'), such that $\mathcal{A}_{rel} \models F$ holds if and only if $\mathcal{A} \models F'$. Under the assumption that $\text{Th}(\mathcal{A})$ is decidable, we then obtain the following decision algorithm for $\text{Th}(\mathcal{A}_{rel})$ which checks whether $F \in \text{Th}(\mathcal{A}_{rel})$:

(i) Construct the formula F' from F, such that $\mathcal{A}_{rel} \models F$ holds if and only if $\mathcal{A} \models F'$,

(ii) Use the decision algorithm for $\text{Th}(\mathcal{A})$ in order to check whether $F' \in \text{Th}(\mathcal{A})$,

(iii) Output „yes", if $F' \in \text{Th}(\mathcal{A})$ and „no" otherwise.

Let F be a formula, such that $\mathcal{A}_{rel}$ is suitable for F. Every atomic formula in F is of the form $R(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are variables (as F does not contain any function symbols). If R corresponds to a predicate symbol in $\mathcal{A}$, then we leave $R(x_1, \ldots, x_n)$ in F' as it is. If R corresponds to a function symbol f in $\mathcal{A}$, then we replace $R(x_1, \ldots, x_n)$ in F' by $x_n = f(x_1, \ldots, x_{n-1})$. By definition of $\mathcal{A}_{rel}$, we find that $\mathcal{A}_{rel} \models F$ holds if and only if $\mathcal{A} \models F'$.

Task 2
Let $(\mathbb{N}, +, f)$ be a structure, where

- $\mathbb{N}$ denotes the universe of the structure,
- $+$ denotes a binary function symbol interpreted as the addition of natural numbers, and
- f denotes a unary function symbol interpreted as the function $f\colon \mathbb{N} \to \mathbb{N}$ with $f(x) = x^2$.

Show that $\text{Th}(\mathbb{N}, +, f)$ is undecidable.

Solution:
Let F be a formula such that $(\mathbb{N}, +, \cdot)$ is suitable for F. We transform F into a formula F' such that $(\mathbb{N}, +, f)$ is suitable for F', and such that $F \in \text{Th}(\mathbb{N}, +, \cdot)$ if and only if $F' \in \text{Th}(\mathbb{N}, +, f)$. Let $\mathcal{A}_{rel} = (\mathbb{N}, P_1, P_2)$ denote the relational structure corresponding to $(\mathbb{N}, +, \cdot)$ with

$$\begin{aligned} P_1 &= \{(a_1, a_2, a) \in \mathbb{N}^3 \mid a_1 + a_2 = a\} \quad \text{and} \\ P_2 &= \{(a_1, a_2, a) \in \mathbb{N}^3 \mid a_1 \cdot a_2 = a\}. \end{aligned}$$

Using the construction from the lecture (slide 33), we transform F into a formula F'', such that $\mathcal{A}_{rel}$ is suitable for F'' and $F \in \text{Th}(\mathbb{N}, +, \cdot)$ if and only if $F'' \in \text{Th}(\mathcal{A}_{rel})$. Every atomic formula in F'' is of the form $P_1(x_1, x_2, x_3)$, $P_2(x_1, x_2, x_3)$ (or $x_1 = x_2$), where x_1, x_2, x_3 are variables. We find that $z = x \cdot y$ can be defined in $(\mathbb{N}, +, f)$ as

$$z = x \cdot y \quad := \quad f(x) + f(y) + z + z = f(x + y).$$

In order to obtain F', replace every occurrence of $P_1(x_1, x_2, x)$ in F'' by $x_1 + x_2 = x$ and every occurrence of $P_2(x_1, x_2, x)$ in F'' by $f(x_1) + f(x_2) + x + x = f(x_1 + x_2)$: This yields a formula F', such that $(\mathbb{N}, +, f)$ is suitable for F' and such that $F \in \text{Th}(\mathbb{N}, +, \cdot)$ if and only if $F' \in \text{Th}(\mathbb{N}, +, f)$. If $\text{Th}(\mathbb{N}, +, f)$ were decidable, we could obtain a decision algorithm for $\text{Th}(\mathbb{N}, +, \cdot)$ by transforming a formula F (for which $(\mathbb{N}, +, \cdot)$ is suitable) into a formula F' (for which is $(\mathbb{N}, +, f)$ suitable), such that $F \in \text{Th}(\mathbb{N}, +, \cdot)$ if and only if $F' \in \text{Th}(\mathbb{N}, +, f)$, and then use the decision algorithm for $\text{Th}(\mathbb{N}, +, f)$ to check whether $F' \in \text{Th}(\mathbb{N}, +, f)$. As $\text{Th}(\mathbb{N}, +, \cdot)$ is undecidable by Gödel's Theorem, $\text{Th}(\mathbb{N}, +, f)$ is undecidable as well.

Task 3
Let $(\mathbb{Z}, +, \cdot)$ be a structure, where

- $\mathbb{Z}$ denotes the universe of the structure,
- $+$ denotes a binary function symbol interpreted as the addition of integers, and
- $\cdot$ denotes a binary function symbol interpreted as the multiplication of integers.

Show that $\text{Th}(\mathbb{Z}, +, \cdot)$ is undecidable.
Hint: Apply Lagrange's four-square theorem:

Theorem 1 (Lagrange's four-square theorem)
Every natural number can be represented as the sum of four integer squares, that is, for every $x \in \mathbb{N}$, there are integers $x_1, x_2, x_3, x_4 \in \mathbb{Z}$, such that $x = x_1^2 + x_2^2 + x_3^2 + x_4^2$.

Solution:
Note that Lagrange's four-square theorem does not hold for negative integers. Let F be a formula such that $(\mathbb{N}, +, \cdot)$ is suitable for F. We transform F into a formula F', such that

$(\mathbb{Z}, +, \cdot)$ is suitable for F', and such that $F \in \mathrm{Th}(\mathbb{N}, +, \cdot)$ if and only if $F' \in \mathrm{Th}(\mathbb{Z}, +, \cdot)$. We define

$$\mathrm{natural}(x) := \exists y_1 \exists y_2 \exists y_3 \exists y_4 (x = y_1 \cdot y_1 + y_2 \cdot y_2 + y_3 \cdot y_3 + y_4 \cdot y_4).$$

In order to obtain F', we replace every occurrence of a subformula $\exists x_i G$ by

$$\exists x_i \left(G \wedge \mathrm{natural}(x_i)\right)$$

and every occurrence of a subformula $\forall x_i G$ by

$$\forall x_i \left(\mathrm{natural}(x_i) \rightarrow G\right).$$

This yields a formula F', such that $(\mathbb{Z}, +, \cdot)$ is suitable for F' and such that $F \in \mathrm{Th}(\mathbb{N}, +, \cdot)$ if and only if $F' \in \mathrm{Th}(\mathbb{Z}, +, \cdot)$. By Gödel's Theorem, $\mathrm{Th}(\mathbb{N}, +, \cdot)$ is undecidable. Thus, $\mathrm{Th}(\mathbb{Z}, +, \cdot)$ is undecidable as well.

Task 4
Consider the structure $(\mathbb{N}, +, \cdot, s, 0)$. Use Gödel's β-function in order to formalize the following statements in predicate logic:

(a) $x^y = z$ (use free variables x, y and z),

(b) Fermat's Last Theorem,

(c) Collatz conjecture.

Solution:
We give the main ideas:

(a) We express $x^y = z$ as: There is a sequence $(a_1, \dots, a_y, a_{y+1})$ with $a_1 = 1$, $a_{i+1} = a_i \cdot x$ for $1 \leq i \leq y$ and $a_{y+1} = z$. This holds if there are $t, p \in \mathbb{N}$ with $\beta(t, p, 1) = 1$, $\beta(t, p, i + 1) = \beta(t, p, i) \cdot x$ for every $1 \leq i \leq y$ and $\beta(t, p, y + 1) = z$.

(b) Fermat's Last Theorem states the following: For all natural numbers $a, b, c \geq 1$ and $n \geq 3$ we have $a^n + b^n \neq c^n$. We already know how to formalize $x^y = z$. From Exercise 2, Task 2, we know how to formalize the numbers 1 and 2 and the relations $\geq$ and $>$ in $(\mathbb{N}, +, \cdot, s, 0)$. We can thus formalize: If $a, b, c \geq 1$ and $n > 2$ and $a^n = a'$, $b^n = b'$ and $c^n = c'$, then $a' + b' \neq c'$.

(c) Let $f \colon \mathbb{N} \rightarrow \mathbb{N}$ be defined as $f(2n) = n$ and $f(2n + 1) = 3(2n + 1) + 1$. Let C_n be the sequence $(n, f(n), f(f(n)), \dots)$. We write $C_n[i]$ for the ith element of the sequence. The Collatz conjecture is the following question: Is there for every n an integer j, such that $C_n[j] = 1$? The function f can be formalized by distinguishing between odd and even numbers and by defining the numbers 2 and 3 (Exercise 2, Task 2). Using the β-function, we can formalize the Collatz conjecture as follows: For every $n \in \mathbb{N}$ there are $t, p \in \mathbb{N}$, such that $\beta(t, p, 1) = n$, $\beta(t, p, i + 1) = f(\beta(t, p, i))$ and there is $j \in \mathbb{N}$ such that $\beta(t, p, j) = 1$.