Exercise 5

Task 1
Compute the following convolutions of words:

(a) $ababab \otimes aaabbbaaa$,

(b) $a \otimes babab \otimes bab \otimes aaaaaa$,

(c) $w_1 \otimes w_2 \otimes \cdots \otimes w_n$ for $n \geq 1$, where $w_k = (ab)^k$,

(d) $(aba \otimes baba) \otimes (bab \otimes abab)$.

Solution: (a) $ababab \otimes aaabbbaaa = (a,a)(b,a)(a,a)(b,b)(a,b)(b,b)(\#,a)(\#,a)(\#,a)$

(b) $(a,b,b,a)(\#,a,a,a)(\#,b,b,a)(\#,a,\#,a)(\#,b,\#,a)(\#,\#,\#,a)$

(c) $(a,a,\ldots,a)(b,b,\ldots,b)(\#,a,\ldots,a)(\#,b,\ldots,b)\cdots(\#,\ldots,\#,a)(\#,\ldots,\#,b)$

(d) $aba \otimes baba = (a,b)(b,a)(a,b)(\#,a)$ and $bab \otimes abab = (b,a)(a,b)(b,a)(\#,b)$ Thus, we obtain $((a,b),(b,a))((b,a),(a,b))((a,b),(b,a))((\#,a),(\#,b))$

Task 2
Let $\Sigma = \{a,b\}$. We consider the following relations on Σ^*:

(a) *Equality* $=$, that is,

$$u = v \iff u \text{ is equal to } v,$$

(b) the *lexicographic order* $\leq_{\mathsf{lex}}$ defined by

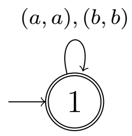
$$u \leq_{\mathsf{lex}} v \iff u \text{ is a prefix of } v \text{ or}$$
$$\text{there are } x,y,z \in \Sigma^* \text{ such that } u = xay \text{ and } v = xbz,$$

(c) the *length-lexicographic order* $\leq_{\mathsf{llex}}$ is defined by

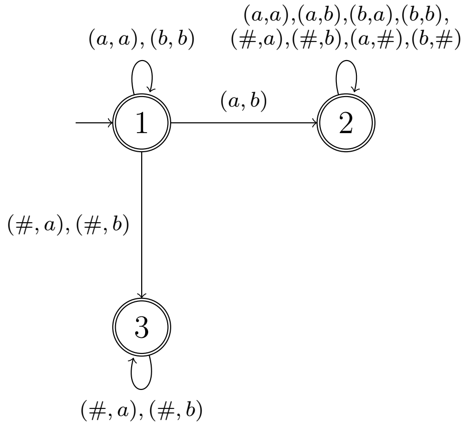
$$u \leq_{\mathsf{llex}} v \iff |u| < |v| \text{ or } (|u| = |v| \text{ and } u \leq_{\mathsf{lex}} v).$$

Show that the relations are synchronously rational.

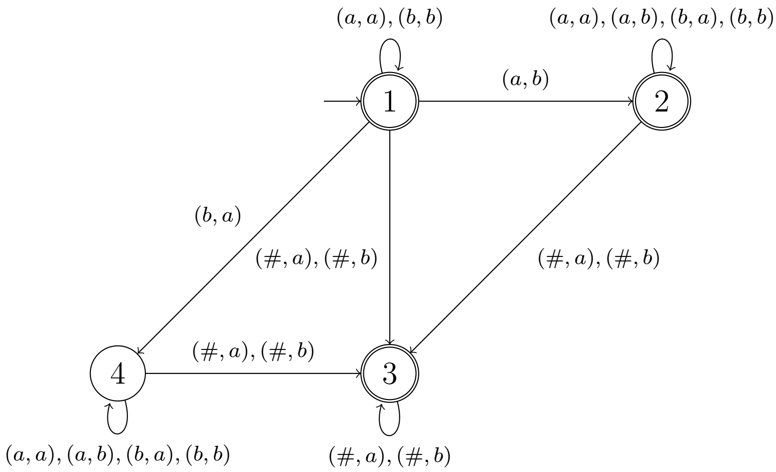
Solution:
(a) A synchronous 2-tape automaton for *equality*:



(b) A synchronous 2-tape automaton for the *lexicographic order* $\leq_{\mathsf{lex}}$:



We stay in state 1 as long as the prefixes of both strings that we have already read are identical. We transition into state 3, if the first string is shorter than the second string, that is, the first string is a proper prefix of the second string (and thus, we accept). We transition from state 1 into state 2, if we read (a, b) and then stay in state 2: In this case, the first string can be written as xay and the second string can be written as xbz for $x, y, z \in \Sigma^*$ (and hence, we accept).
Note that strings which do not belong to $\{w_1 \otimes w_2 \mid w_1, w_2 \in \Sigma^*\}$ are ignored.
(c) A synchronous 2-tape automaton for the *length-lexicographic order* $\leq_{\mathsf{llex}}$:



We stay in state 1 as long as the prefixes of both strings that we have already seen are identical. We transition into state 3, if the first string is shorter than the second string: In this case, the first string is a proper prefix of the second string. We transition from state 1 into state 2, if we read (a, b). We stay in state 2 as long as we read further characters of both strings, that is, in this case, the strings are of the same length and the first string can be written as xay and the second string can be written as xbz for $x, y, z \in \Sigma^*$ (and hence, we accept). We transition from state 2 into state 3, if the first string is shorter than the second string (and accept). We transition from state 1 into state 4 if (b, a) is read, and stay in this state as long as we keep reading characters from both strings. In this case, both strings are of the same lenght, but the second string can be written as xay and the first string can be written as xbz for $x, y, z \in \Sigma^*$ (and thus, we do not accept). We transition from state 3 into state 4, if the first string is shorter than the second string (and accept).

Task 3
Let $\Sigma = \{a, b\}$ and let $n \geq 1$. Show that the language

$$\{w_1 \otimes \cdots \otimes w_n \mid w_1, \ldots, w_n \in \Sigma^*\} \subseteq (\Sigma_\#^n)^*$$

is regular by constructing a finite automaton for this language.

Solution:
Let $M = (Z, \Sigma_\#^n, \delta, S, E)$ denote the (non-deterministic) finite automaton which accepts this language, where Z denotes the set of states, S denotes the set of initial states, E denotes the set of accept states and δ denotes the transition function. Let

$$Z = \{(i_1, \ldots, i_n) \mid i_k \in \{0, 1\} \text{ for every } k\} \setminus \{(0, \ldots, 0)\}.$$

The idea is that the k-th element of a state $(i_1, \ldots, i_n)$ satisfies $i_k = 0$ if the k-th word w_k in the convolution has already ended and we have read a # symbol in the k-th component of a tuple (and thus, we are only allowed to read # in this component) or $i_k = 1$ if we still read characters from the alphabet Σ in this component. We define the set of initial states as $S = \{(1, 1, \ldots, 1)\}$ and the set of accept states as $E = Z$. Moreover, we define the transition function δ as follows: Let $(i_1, \ldots, i_n) \in Z$ and $(a_1, \ldots, a_n) \in \Sigma_\#^n$. We set

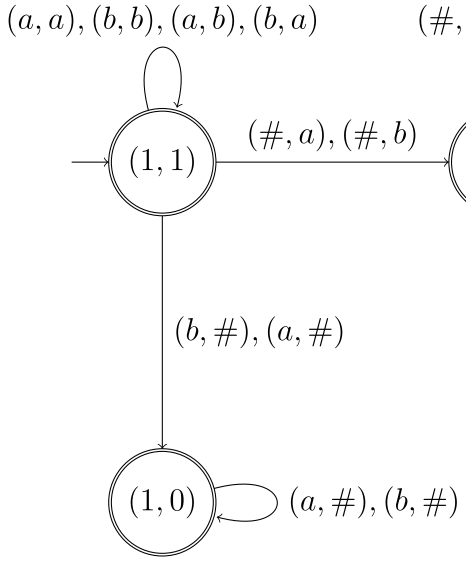
$$\delta((i_1, \ldots, i_n), (a_1, \ldots, a_n)) = \emptyset,$$

if there is an index $1 \leq k \leq n$, such that $i_k = 0$ and $a_k \in \Sigma$ or if $a_k = \#$ for all $1 \leq k \leq n$. Otherwise, we set

$$\delta((i_1, \ldots, i_n), (a_1, \ldots, a_n)) = \{(j_1, \ldots, j_n)\},$$

where $j_k = 1$ if $i_k = 1$ and $a_k \in \Sigma$, $j_k = 0$ if $i_k = 1$ and $a_k = \#$, and $j_k = 0$ if $i_k = 0$ and $a_k = \#$.

For example, if $n = 2$, the automaton looks like this:


$(a,a),(b,b),(a,b),(b,a)$
$(\#,a),(\#,b)$
$(1,1)$
$(\#,a),(\#,b)$
$(0,1)$
$(b,\#),(a,\#)$
$(1,0)$
$(a,\#),(b,\#)$