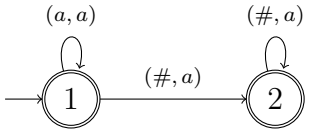
Universität Siegen
Lehrstuhl Theoretische Informatik
Markus Lohrey

Exercise 7

Task 1
Check whether $(\mathbb{N}, \leq) \models \exists x \forall y (x \leq y)$ holds by applying the technique from the proof of the Theorem of Khoussainov and Nerode.

Solution:
In the solution of exercise sheet 7, task 1, we showed that $(\mathbb{N}, \leq)$ is isomorphic to the following automatic structure: $(\{a\}^*, \leq_a)$, where $a^i \leq_a a^j$ if and only if $i \leq j$. In particular,



is a 2-tape automaton for $\leq_a$.

As a first step, we transform the formula $\exists x \forall y (x \leq_a y)$ into an equivalent formula which does not contain a $\forall$-quantifier: A $\forall$-quantifier can be expressed using a negation of an $\exists$-quantifier. We find

$$\exists x \forall y (x \leq_a y) \equiv \exists x \neg \exists y \neg (x \leq_a y).$$

Now let $F = \exists x \neg \exists y \neg (x \leq_a y)$. We start with the (atomic) subformula $F_1 = x \leq_a y$, which is treated in case 1 (slide 75) in the proof of the Theorem of Khoussainov und Nerode. This means that we construct a synchronous 2-tape automaton B_{F_1}, such that

$$K(B_{F_1}) = \{(w_1, w_2) \in \{a\}^* \times \{a\}^* \mid w_1 \leq_a w_2\}.$$

In this concrete example, we can take the 2-tape automaton from above, which accepts precisely this relation. Note that all variables of F are free variables in F_1, and we assume that they are ordered according to their occurrence in F_1 – thus, the homomorphism from case 1 on slide 75 is the identity mapping.

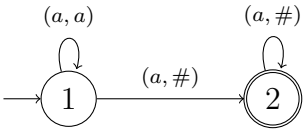
Next, we consider the subformula $F_2 = \neg F_1 = \neg (x \leq_a y)$: This corresponds to case 3 (slide 77) from the proof of the Theorem of Khoussainov and Nerode. We thus need a 2-tape automaton B_{F_2}, such that

$$L(B_{F_2}) = \{w_1 \otimes w_2 \mid w_1, w_2 \in \{a\}^*\} \setminus L(B_{F_1}),$$

respectively,

$$K(B_{F_2}) = \{(a^n, a^m) \mid n > m\}.$$

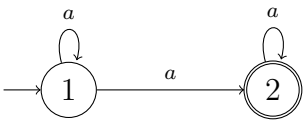
The following automaton satisfies this property:



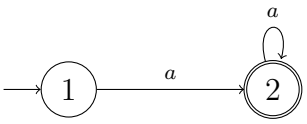
Next, we consider the subformula $F_3 = \exists y F_2 = \exists y \neg(x \leq_a y)$. This corresponds to case 5 (slide 78). Let f be the homomorphism defined by $f(w_1 \otimes w_2) = w_1$ (slide 78). We are looking for an automaton B_{F_3}, such that

$$L(B_{F_3}) = f(L(B_{F_2})).$$

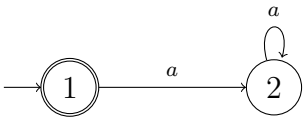
This means that we simply ignore the second component:



This non-deterministic automaton accepts the same language as the following deterministic automaton:



Next, we consider the subformula $F_4 = \neg F_3 = \neg\exists y \neg(x \leq_a y)$, which again corresponds to case 3 (slide 77). It is easy to see that the complement of $L(B_{F_3})$ only contains the empty word ε. Hence, B_{F_4} is the following automaton (which is obtained by switching accept and non-accept states in the above deterministic automaton):



The formula F is of the form $F = \exists x F_4$ (as on slide 80). We have $(\{a\}^*, \leq_a) \models F$ if and only if $L(B_{F_4}) \neq \emptyset$. As $L(B_{F_4}) = \{\varepsilon\}$, we find that $F \in \mathrm{Th}(\mathbb{N}, \leq)$.

Task 2
Prove or disprove the following statement: If $(\mathbb{N}, R_1)$ and $(\mathbb{N}, R_2)$ are automatically presentable, then $(\mathbb{N}, R_1, R_2)$ is automatically presentable.

Solution:
The statement is not correct: By exercise sheet 7, task 1, we know that $(\mathbb{N}, \leq)$ and $(\mathbb{N}, M)$ are automatically presentable, where $M \subseteq \mathbb{N}$ is a unary relation. However, we show that $(\mathbb{N}, \leq, M)$ is not necessarily automatically presentable: We can define every natural number $n \in \mathbb{N}$ using $\leq$ (and $=$): Let a and b be free variables. Define $a < b$ by $a \leq b \wedge \neg(a = b)$. We define the following formulas:

$$\begin{aligned} s(a,b) &= \neg\exists z(a < z \wedge z < b) \wedge (a < b), \\ 0(a) &= \neg\exists z\ z < a. \end{aligned}$$

The formula $s(a, b)$ states that there is no natural number which is greater than a and smaller than b (that is, b is the immediate successor of a). The formula $0(a)$ defines the natural number 0, as there is no natural number, which is smaller than 0. Furthermore, we define $s^0(a) = 0(a)$ and for every $i \in \mathbb{N}$ let

$$s^{i+1}(a) = \exists x_i(s^i(x_i) \wedge s(x_i, a)).$$

Then $s^n(a)$ states that the free variable a is the natural number n. Let M be an undecidable subset of $\mathbb{N}$. If the structure $(\mathbb{N}, \leq, M)$ were automatically presentable, then $\mathrm{Th}(\mathbb{N}, \leq, M)$ would be decidable by the Theorem of Khoussainov/Nerode. Then we could check if $n \in M$, by checking if $\forall x(s^n(x) \rightarrow M(x)) \in \mathrm{Th}(\mathbb{N}, \leq, M)$. As M is undecidable, we obtain a contradiction. Thus, $(\mathbb{N}, \leq, M)$ cannot be automatically presentable.