

Exercise 8

Task 1

Consider the structure $(\mathbb{N}, 0, s)$, where s is the successor function ($s(n) = n + 1$). Formulate the axiom of induction using an MSO-sentence.

Axiom of induction: Every subset of the natural numbers, which contains 0 and which contains for every element of the subset also its successor, is equal to the set of natural numbers.

Task 2

Consider the structure $(\mathbb{R}, <)$. Formulate the following statements using MSO-sentences:

- (a) Every set is a subset of itself.
- (b) There is a non-empty set.
- (c) For every set X there is a set Y , such that the intersection of X and Y is empty.
- (d) Every set that contains at least two distinct elements has a proper subset.
- (e) Every proper bounded open interval contains a proper closed subinterval.
- (f) There is a set X , such that for every set Y the union of X and Y equals Y

Task 3

Let $\mathcal{G} = (V, E)$ be a directed graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. We consider $\mathcal{G}$ as a structure with universe V and binary relation E . Formulate the following statements as MSO-formulas:

- (a) The graph is strongly connected.
- (b) The graph is bipartite (= the underlying undirected graph is bipartite).
- (c) The graph is a tree with a root.

Task 4

Find MSO-formulas for the following regular languages:

- (a) $L_1 = L((a|b)^*a)$
- (b) $L_2 = \{w \in \Sigma^+ \mid w \text{ begins and ends with } b\}$
- (c) $L_3 = L(b(a|b)^*b)$