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Exercise 8

Task 1
Consider the structure $(\mathbb{N}, 0, s)$, where s is the successor function ($s(n) = n+1$). Formulate the axiom of induction using an MSO-sentence.

<u>Axiom of induction</u>: Every subset of the natural numbers, which constains 0 and which contains for every element of the subset also its successor, is equal to the set of natural numbers.

Solution:

$$\forall S((0 \in S \land \forall x(x \in S \to s(x) \in S)) \to \forall x(x \in S))$$

Task 2
Consider the structure $(\mathbb{R}, <)$. Formulate the following statements using MSO-sentences:

(a) Every set is a subset of itself.

(b) There is a non-empty set.

(c) For every set X there is a set Y, such that the intersection of X and Y is empty.

(d) Every set that contains at least two distinct elements has a proper subset.

(e) Every proper bounded open interval contains a proper closed subinterval.

(f) There is a set X, such that for every set Y the union of X and Y equals Y

Solution: (a) We first define $X \subseteq Y$ as

$$\forall x(x \in X \to x \in Y).$$

The formula for part (a) is now obtained as

$$\forall X(X \subseteq X).$$

(b) $\exists X \exists x(x \in X)$

(c) $\forall X \exists Y \neg \exists x(x \in X \land x \in Y)$

(d) $\forall X((\exists x \exists y((x \in X) \land (y \in X) \land (x \neq y))) \to (\exists Y((Y \subseteq X) \land \exists z(z \in X \land z \notin Y))$

(e) We first define closedinterval(X) (for intervals of the form $[a,b]$ with $a < b$) as

$$\begin{aligned}&\exists a\exists b((a < b) \wedge (a \in X) \wedge (b \in X)\\&\wedge \forall x(((x < a) \to \neg(x \in X))\\&\wedge ((a < x) \wedge (x < b) \to (x \in X))\\&\wedge ((b < x) \to \neg(x \in X))).\end{aligned}$$

In the same way, we define openinterval(X) (for bounded intervals of the form (a,b) with $a < b$) as

$$\begin{aligned}&\exists a\exists b((a < b) \wedge \neg(a \in X) \wedge \neg(b \in X)\\&\wedge \forall x(((x < a) \to \neg(x \in X))\\&\wedge ((a < x) \wedge (x < b) \to (x \in X))\\&\wedge ((b < x) \to \neg(x \in X))).\end{aligned}$$

We can now define the statement as

$$\forall X(\text{openinterval}(X) \to \exists Y(\text{closedinterval}(Y) \wedge Y \subseteq X)).$$

(f) $\exists X\forall Y(\forall x((x \in X) \vee (x \in Y)) \to (x \in Y))$

Task 3
Let $\mathcal{G} = (V,E)$ be a directed graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. We consider $\mathcal{G}$ as a structure with universe V and binary relation E. Formulate the following statements as MSO-formulas:

(a) The graph is strongly connected.

(b) The graph is bipartite (= the underlying undirected graph is bipartite).

(c) The graph is a tree with a root.

Solution:
Recall the formula reach for variables x and y (slide 146):

$$\text{reach}(x,y) = \forall X\big((x \in X \wedge \forall u\forall v((u \in X \wedge E(u,v)) \to v \in X)) \to y \in X\big)$$

The formula reach(x,y) expresses that in the graph $\mathcal{G}$ there is a path from x to y.

(a) A graph is strongly connected, if every vertex is reachable from every other vertex:
$\forall x\forall y(x \neq y \to \text{reach}(x,y))$

(b) A graph is bipartite, if its vertices can be divided into two disjoint subsets, such that every edge connects a vertex from one on the sets to a vertex from the other set:
$\exists X\forall x\forall y(E(x,y) \to (x \in X \leftrightarrow y \notin X))$

(c) A tree is a graph without cycles, such that every vertex can be reached from the root, and every vertex except for the root has exactly one incoming edge.

No cycles: $\forall x \forall y((x \neq y) \wedge \mathrm{reach}(x,y)) \rightarrow \neg \mathrm{reach}(y,x)) \wedge \forall z \neg E(z,z)$

Every vertex can be reached from the root and every vertex except for the root has exactly one incoming edge:

$$\exists x \forall y((y \neq x \rightarrow \mathrm{reach}(x,y)) \wedge (y \neq x \rightarrow \exists v(E(v,y) \wedge \forall u((u \neq v) \rightarrow (\neg E(u,y))))))$$

The formula for (c) is obtained by combining the two statements.

Task 4
Find MSO-formulas for the following regular languages:

(a) $L_1 = L\left((a|b)^* a\right)$

(b) $L_2 = \{w \in \Sigma^+ \mid w \text{ begins and ends with } b\}$

(c) $L_3 = L(b(a|b)^* b)$

Solution:

(a) $\exists x(\forall u(u \leq x) \wedge P_a(x))$

(b) $\exists x(\forall u(u \leq x) \wedge P_b(x)) \wedge \exists y(\forall u(y \leq u) \wedge P_b(y))$

(c) $\exists x \exists y(x \neq y \wedge \forall u(u \leq x) \wedge P_b(x) \wedge \forall u(y \leq u) \wedge P_b(y))$