Exercise 1

Task 1. Prove that the *Vandermonde* Matrix

$$V(a_1, \dots, a_n) = \begin{pmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} \end{pmatrix}$$

is invertible if and only if the parameters $a_1, \dots, a_n \in \mathbb{C}$ are pairwise distinct numbers.
Hint: Given a linear combination $\lambda_0 V_0 + \ldots + \lambda_{n-1} V_{n-1} = 0$ of the columns $V_0, \dots, V_{n-1}$ of the Vandermonde Matrix, consider the polynomial $p(z) = \lambda_0 + \lambda_1 z^1 + \ldots + \lambda_{n-1} z^{n-1}$.

Task 2. Show that the n-th roots of a complex number $w = re^{i\theta} \neq 0$ ($r, \theta \in \mathbb{R}$, $r > 0$), i.e., the solutions to the equation $z^n = w$, can be expressed as

$$z = \sqrt[n]{r} \cdot \exp\left(i\left(\tfrac{\theta}{n} + \tfrac{2\pi k}{n}\right)\right).$$

Task 3. Use the FFT algorithm (slides 14-16) to do the following.

(a) Compute the discrete Fourier transformation (slide 10) of the complex polynomial

$$f(z) = z + 2z^2 + 3z^3.$$

(b) Compute the product of complex polynomials $(z + 2) \cdot (2z - 1)$.

Task 4. Let $A, B \subseteq \{1, \dots, n\}$ be sets of size $|A|, |B| \in \Theta(n)$. We want to compute

$$C := \{a + b : a \in A, b \in B\}$$

as well as the number of ways to write each $c \in C$ as a sum of elements in A and B. Specify an algorithm to solves this problem in time $\mathcal{O}(n \log n)$.