Exercise 4

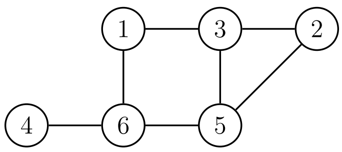
Task 1. Consider the following algorithm which, on input of $n \times n$ matrices A, B, C, decides probabilistically whether $AB = C$ holds.

- Choose a vector $x \in \{0,1\}^n$ uniformly at random.
- Compute the vector $w = A(Bx) - Cx$.
- Accept if $w = 0$ and reject otherwise.

(a) Prove that the algorithm answers correctly with probability at least $\frac{1}{2}$.

(b) Modify the algorithm so that it answers correctly with probability at least $\frac{7}{8}$.

Task 2. Let $G = (V, E)$ be the following undirected graph.



(a) Compute the Tutte matrix T_G (slide 72) and its determinant $\det(T_G)$.

(b) Find all perfect matchings of the graph G.

Task 3. Let $G = (U \,\dot\cup\, V, E)$ be an undirected bipartite graph, i.e., every edge $\{u, v\} \in E$ has exactly one endpoint in U and exactly one in V. Suppose further that the graph G is balanced in that $|U| = |V|$, and let $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$.

The *Edmonds matrix* of the graph G is the matrix $A_G = (A_{ij})$ given by

$$A_{ij} = \begin{cases} x_{ij} & \text{if } \{u_i, v_j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that $\det(A_G) \neq 0$ if, and only if, G has a perfect matching.

(b) Conclude that $\mathrm{rk}(A_G)$, the rank of A_G, is the largest size of a matching of G.

Hint: The rank of a matrix is the largest order of a non-zero minor.