Exercise 3

Task 1. Determine whether or not the following states are entangled.

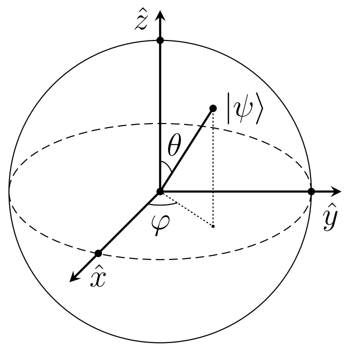
(a) $\frac{1}{\sqrt{2}}\big(|000\rangle + |111\rangle \big)$ (b) $\frac{1}{\sqrt{3}}\big(|001\rangle + |010\rangle + |100\rangle \big)$ (c) $\frac{1}{\sqrt{2}}\big(|00\rangle - i\,|10\rangle \big)$

Task 2. Show that $|x\rangle\langle x| = |y\rangle\langle y|$ holds if, and only if, $|x\rangle = e^{i\gamma}|y\rangle$ for some γ.

Task 3. Let $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ be a pure state of a single-qubit system. Show that, for suitable angles θ and φ as well as a global phase $e^{i\gamma}$, one can write the state as

$$|\psi\rangle = e^{i\gamma}\left(\cos\tfrac{\theta}{2}|0\rangle + e^{i\varphi}\sin\tfrac{\theta}{2}|1\rangle\right).$$

Note: The global phase $e^{i\gamma}$ is without physical meaning. The remaing information, which is encoded in the angles θ and φ, realizes a point on the *Bloch sphere*.



Task 4. Let A and B be unitary matrices corresponding to single-qubit quantum gates. We can then consider the quantum gate defined by their direct sum

$$A \oplus B = \begin{pmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ 0 & 0 & B_{11} & B_{12} \\ 0 & 0 & B_{21} & B_{22} \end{pmatrix}.$$

(a) Show that the controlled operation applied by this gate is as follows.

- If the first qubit is in state $|0\rangle$, then A is applied to the second qubit.
- If the first qubit is in state $|1\rangle$, then B is applied to the second qubit.

(b) Draw the quantum circuit corresponding to this gate.

Convention: Operations controlled by a qubit being in state $|1\rangle$ are drawn with a *solid* vertex (as is shown on slides 73 and 74). When controlled by a qubit being in state $|0\rangle$, an *empty* vertex is used instead. The following shows both cases.