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# Exercise 4

**Task 1.** Let $\rho$ be a density matrix. The measurement $B = \{\Pi_1, ..., \Pi_k\}$ transforms the density matrix $\rho$ into $\Sigma_{i=1}^k \Pi_i \rho \Pi_i$. The outcome $i \in [1, k]$ occurs with probability $\mathrm{tr}(\Pi_i \rho)$ and leads to the collapsed density matrix

$$\rho' = \frac{\Pi_i \rho \Pi_i}{\mathrm{tr}(\Pi_i \rho \Pi_i)}. \tag{1}$$

This formalism also encapsulates the measurement of a state $|\psi\rangle$. To see this, let $|\psi'\rangle$ denote the post-measurement state of our system upon obtaining outcome $i$.

(a) Give an explicit formula for $|\psi'\rangle$. (Don't forget to renoromalize.)

(b) Show that, for $\rho = |\psi\rangle \langle\psi|$, equation (1) yields $\rho' = |\psi'\rangle \langle\psi'|$.

**Task 2.** Let $A = \{\Pi_0, \Pi_1\}$ be the set consisting of the operators

$$\Pi_0 = |00\rangle \langle 00| + |01\rangle \langle 01| \quad \text{and} \quad \Pi_1 = |10\rangle \langle 10| + |11\rangle \langle 11|.$$

(a) Show that $A$ is a projective measurement (slide 55). What does it measure?

Consider a quantum state $|\psi\rangle = \alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{10} |10\rangle + \alpha_{11} |11\rangle$.

(b) Calculate the probability of measuring the first qubit of $|\psi\rangle$ in state $|0\rangle$.

(c) What is the resulting state $|\psi'\rangle$ after the measurement?

**Task 3.** The *Toffoli* gate, also known as the *controlled-controlled-not* (CCNOT) gate, is the 3-qubit gate inverting the third qubit if, and only if, the first two qubits are set. I.e.,

$$\mathrm{CCNOT}\,|110\rangle = |111\rangle\,, \quad \mathrm{CCNOT}\,|111\rangle = |110\rangle\,,$$

and all other basis states are preserved. Verify, on $|110\rangle$ and $|111\rangle$, that the following quantum circuit implements the Toffoli gate, wherein $H$ and $T$ denote the Hadamard and the $\frac{\pi}{8}$ gate, respectively, and $S = T^2$ denotes the so-called phase gate (slide 86).