Universität Siegen
Lehrstuhl Theoretische Informatik
Markus Lohrey

Exercise 11

Task 1
Let $f\colon \mathbb{R} \to \mathbb{R}$ be a polynomial defined by

$$f(x) = 4x^5 - 2x^4 + 25x^2 - 5x + 1.$$

Use Cauchy's bound to find an interval which contains all real-valued zeros of f.

Solution:
All real roots of the polynomial f belong to an interval $(-c, c)$, where

$$c := 1 + \frac{\max\{1, 5, 25, 2\}}{4} = 7.25.$$

Task 2
Consider the structure $(\mathbb{N}, 0, s)$, where s is the successor function ($s(n) = n+1$). Formulate the axiom of induction using an MSO-sentence!

Axiom of induction: Every subset of the natural numbers, which constains 0 and which contains for every element of the subset also its successor, is equal to the set of natural numbers.

Solution:

$$\forall S((0 \in S \land \forall x(x \in S \to s(x) \in S)) \to \forall x(x \in S))$$

Task 3
Consider the structure $(\mathbb{R}, <)$. Formulate the following statements using MSO-sentences:

(a) Every set is a subset of itself.

(b) Every open interval contains a closed subinterval.

Solution: (a) We first define $X \subseteq Y$ as

$$\forall x(x \in X \to x \in Y).$$

The formula for part (a) is now obtained as

$$\forall X(X \subseteq X).$$

(b) We first define interval(X) (for finite intervals) as

$$\exists a \exists b \forall x (((x < a) \rightarrow \neg(x \in X)) \wedge ((a < x) \wedge (x < b) \rightarrow (x \in X)) \wedge ((b < x) \rightarrow \neg(x \in X)).$$

(In the same way, we could also check whether X is an unbounded interval). An open interval does not contain a smallest or largest number. We thus define openinterval(X) as

$$\text{interval}(X) \wedge \forall x (x \in X \rightarrow (\exists y (y \in X \wedge x < y) \wedge \exists z (z \in X \wedge (z < x)))).$$

In a similar way, we define closedinterval(X) as

$$\begin{aligned} \text{interval}(X) \wedge \exists a (a \in X \wedge \forall y (y \in X \rightarrow (a < y \vee y = a))) \\ \wedge \exists b (b \in X \wedge \forall y (y \in X \rightarrow (y < b \vee y = b))). \end{aligned}$$

We can now define the statement for part (b) as

$$\forall X (\text{openinterval}(X) \rightarrow \exists Y (\text{closedinterval}(Y) \wedge Y \subseteq X)).$$