Exercise 12

Task 1
Let $\mathcal{G} = (V, E)$ be a graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. We consider $\mathcal{G}$ as a structure with universe V and binary relation E. Formulate the following statements as MSO-formulas:

(a) The graph is strongly connected.

(b) The graph is bipartite (= the underlying undirected graph is bipartite).

(c) The graph is a tree with a root.

Solution:
Recall the formula reach for variables x and y (slide 146):

$$\text{reach}(x, y) = \forall X\big((x \in X \wedge \forall u \forall v((u \in X \wedge E(u, v)) \to v \in X)) \to y \in X\big)$$

The formula reach(x, y) expresses that in the graph $\mathcal{G}$ there is a path from x to y.

(a) A graph is strongly connected, if every vertex is reachable from every other vertex:
$\forall x \forall y(x \neq y \to \text{reach}(x, y))$

(b) A graph is bipartite, if its vertices can be divided into two disjoint subsets, such that every edge connects a vertex from one on the sets to a vertex from the other set:
$\exists X \forall x \forall y(E(x, y) \to (x \in X \leftrightarrow y \notin X))$

(c) A tree is a graph without cycles, such that every vertex can be reached from the root, and every vertex except for the root has exactly one incoming edge.

No cycles: $\forall x \neg \text{reach}(x, x)$

Every vertex can be reached from the root and every vertex except for the root has exactly one incoming edge:

$$\exists x \forall y((y \neq x \to \text{reach}(x, y)) \wedge (y \neq x \to \exists v(E(v, y) \wedge \forall u((u \neq v) \to (\neg E(u, y))))))$$

The formula for (c) is obtained by combining the two statements.

Task 2
Find MSO-formulas for the following regular languages:

(a) $L_1 = L\left((a|b)^*a\right)$

(b) $L_2 = \{w \in \Sigma^+ \mid w \text{ begins and ends with } b\}$

(c) $L_3 = L(b(a|b)^*b)$

Solution:

(a) $\exists x(\forall u(u \leq x) \wedge P_a(x))$

(b) $\exists x(\forall u(u \leq x) \wedge P_b(x)) \wedge \exists y(\forall u(y \leq u) \wedge P_b(y))$

(c) $\exists x \exists y(x \neq y \wedge \forall u(u \leq x) \wedge P_b(x) \wedge \forall u(y \leq u) \wedge P_b(y))$