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Exercise 13

Task 1
Find an MSO-formula corresponding to the regular language

$$L = \{w \in \Sigma^+ \mid \text{The number of } a\text{'s in } w \text{ is odd}\}.$$

Hint: Consider the technique from the proof of Büchi's Theorem.

Solution:

(a) For simplicity, let $\Sigma = \{a, b\}$. The automaton $(\{1,2\}, \{a,b\}, \delta, 1, \{2\})$ with

$$\begin{aligned}\delta(1,a) &= 2\\ \delta(1,b) &= 1\\ \delta(2,a) &= 1\\ \delta(2,b) &= 2\end{aligned}$$

accepts the language L. Using Büchi's Theorem, we obtain the following formula:

$$\begin{aligned}
&\exists X_1 \exists X_2 (X_1 \cap X_2 = \emptyset \wedge \forall x (x \in X_1 \vee x \in X_2)\\
&\wedge \exists x (\forall y (x \leq y) \wedge ((P_a(x) \wedge x \in X_2) \vee (P_b(x) \wedge x \in X_1)))\\
&\wedge \exists x (\forall y (y \leq x) \wedge x \in X_2)\\
&\wedge \forall x \forall y (y = x + 1 \to (x \in X_1 \wedge P_a(y) \wedge y \in X_2)\\
&\qquad \vee (x \in X_1 \wedge P_b(y) \wedge y \in X_1)\\
&\qquad \vee (x \in X_2 \wedge P_a(y) \wedge y \in X_1)\\
&\qquad \vee (x \in X_2 \wedge P_b(y) \wedge y \in X_2)))
\end{aligned}$$

Task 2

Which regular languages over $\Sigma = \{a, b, c\}$ correspond to the following MSO formulas?

(a) $\forall x \forall y (P_a(x) \wedge P_b(y) \wedge (x < y) \wedge (\forall z (x < z < y) \rightarrow \neg P_b(z)))$
$\rightarrow (\exists x_1 \exists x_2 (x < x_1 < x_2 < y) \wedge P_c(x_1) \wedge P_c(x_2))$

(b) $\exists X (\exists x \exists y (\forall u (x \leq u \leq y) \wedge x \in X \wedge y \in X) \wedge$
$\forall x \forall y (y = x + 1 \rightarrow (x \in X \leftrightarrow \neg (y \in X))))$

Solution:

(a) A word w from the regular language corresponding to the MSO-formula satisfies: There are at least two occurencences of the character c between an occurrence of a and the subsequent occurrence of b in w.

(b) The regular language contains all words of odd length.