Exercise 2

Task 1

Which of the following problems are decidable, and which of them are recursively enumerable?

(a) Checking whether a formula F of predicate logic is neither valid nor unsatisfiable,

(b) Checking whether a formula F of predicate logic which contains a single unary predicate symbol, but no function symbols is satisfiable?

Solution:

(a) This problem is not recursively enumerable (and thus, not decidable). Let us assume that this problem is recursively enumerable (to deduce a contradiction): That is, we assume that there is an algorithm (algorithm 1) which outputs 'yes' on input F, if F is neither valid nor unsatisfiable, and does not terminate otherwise. Furthermore, we already know that the set of valid formulas of predicate logic is recursively enumerable (Corollary of Gilmore's Theorem). That is, there is an algorithm (algorithm 2), which outputs 'yes' on input F, if F is valid, and does not terminate otherwise. Combining these two algorithms gives us an algorithm (algorithm 3) to check whether a given formula F of predicate logic is satisfiable: Run algorithm 1 and algorithm 2 in parallel (both with input F). If

- algorithm 2 terminates: The formula is valid and hence satisfiable. We output 'yes'
- algorithm 1 terminates: The formula is not unsatisfiable, and hence satisfible. We output 'yes'.

Otherwise, if algorithm 1 and 2 do not terminate, F must be unsatisfiable and algorithm 3 does not terminate either. Hence, algorithm 3 terminates, if the input F is satisfiable, and does not terminate otherwise. However, the set of satisfiable formulas of predicate logic is not recursively enumerable (Corollary of Church's theorem), a contradiction.

(b) This problem is decidable. Let P denote the unary predicate symbol which occurs in F. We claim that if F is satisfiable, then there is a model $\mathcal{A} = (\mathcal{U}_\mathcal{A}, \mathcal{I}_\mathcal{A})$ for F, such that $|\mathcal{U}_\mathcal{A}| \leq 2$. The number of structures $\mathcal{A} = (\mathcal{U}_\mathcal{A}, \mathcal{I}_\mathcal{A})$, for which $|\mathcal{U}_\mathcal{A}| \leq 2$ and which interpret only a unary predicate symbol P is finite (except for isomorphisms): Hence, we can test which of these structures are a model for F.

To prove the claim, assume that F is in *negation normal form*, that is, the negation operator $\neg$ is only applied directly to the atomic formula $P(x)$. Every formula is equivalent to a formula in negation normal form.

Let $\mathcal{A} = (\mathcal{U}_\mathcal{A}, \mathcal{I}_\mathcal{A})$ be a model for F with $\mathcal{I}_\mathcal{A}(P) = P^\mathcal{A}$ and $x_1^\mathcal{A} = a_1, \ldots, x_n^\mathcal{A} = a_n$ for free variables $x_1, \ldots, x_n$ occurring in F. We have to show that there is a model $\mathcal{A}' = (\mathcal{U}'_\mathcal{A}, \mathcal{I}'_\mathcal{A})$ for F, such that $|\mathcal{U}'_\mathcal{A}| \leq 2$.

We define a mapping $\varphi : \mathcal{U}_\mathcal{A} \to \{0, 1\}$ by

$$\varphi(a) = \begin{cases} 0 & \text{if } a \notin P^\mathcal{A}, \\ 1 & \text{if } a \in P^\mathcal{A}. \end{cases}$$

Moreover, we define a structure $\mathcal{A}' = (\mathcal{U}'_\mathcal{A}, \mathcal{I}'_\mathcal{A})$ as follows: The universe $\mathcal{U}'_\mathcal{A}$ of $\mathcal{A}'$ is defined as $\mathcal{U}'_\mathcal{A} = \varphi(\mathcal{U}_\mathcal{A}) = \{\varphi(a) \mid a \in \mathcal{U}_\mathcal{A}\}$. The interpretation function $\mathcal{I}'_\mathcal{A}$ is defined by

$$\mathcal{I}'_\mathcal{A}(P) = P^{\mathcal{A}'} = \begin{cases} \{1\} & \text{if } 1 \in \mathcal{U}'_\mathcal{A}, \\ \emptyset & \text{otherwise}, \end{cases}$$

and $x_1^{\mathcal{A}'} = \varphi(a_1), \ldots, x_n^{\mathcal{A}'} = \varphi(a_n)$. We show inductively in the composition of the formula that $\mathcal{A}'$ is a model for F:

- Let $F = P(x_i)$ with $1 \leq i \leq n$. As $\mathcal{A}$ is a model for F, we have $x_i^\mathcal{A} = a_i \in P^\mathcal{A}$. Thus, $\varphi(a_i) = 1$ and $\varphi(a_i) \in P^{\mathcal{A}'} = \{1\}$, and $\mathcal{A}'$ is a model for F.
- Let $F = \neg P(x_i)$ with $1 \leq i \leq n$. As $\mathcal{A}$ is a model for F, we have $x_i^\mathcal{A} = a_i \notin P^\mathcal{A}$. Thus, $\varphi(a_i) = 0$ and $\varphi(a_i) \notin P^{\mathcal{A}'} = \emptyset$, and $\mathcal{A}'$ is a model for F.
- Let $F = G \wedge H$. As $\mathcal{A} \models F$, we have $\mathcal{A} \models G$ and $\mathcal{A} \models H$. By induction hypothesis, we have $\mathcal{A}' \models G$ and $\mathcal{A}' \models H$, and hence $\mathcal{A}' \models F$.
- Let $F = G \vee H$. This case is analogous to the previous case.
- Let $F = \exists x G$. As $\mathcal{A} \models F$, there exists $a \in \mathcal{U}_\mathcal{A}$ with $\mathcal{A}_{[x/a]} \models G$. By induction hypothesis, we have $\mathcal{A}'_{[x/\varphi(a)]} \models G$. As $\varphi(a) \in \mathcal{U}'_\mathcal{A}$, we find that $\mathcal{A}' \models F$.
- Let $F = \forall x G$. As $\mathcal{A} \models F$, we find that for all $a \in \mathcal{U}_\mathcal{A}$, $\mathcal{A}_{[x/a]} \models G$. By induction hypothesis, we find that $\mathcal{A}'_{[x/\varphi(a)]} \models G$ for all $a \in \mathcal{U}_\mathcal{A}$, respectively, for all $\varphi(a) \in \varphi(\mathcal{U}_\mathcal{A}) = \mathcal{U}'_\mathcal{A}$ (as φ is surjective). Thus, $\mathcal{A}' \models \forall x G$.

Task 2

Let $(\mathbb{N}, +, \cdot)$ be a structure, where

- $\mathbb{N}$ denotes the universe of the structure,
- $+$ und $\cdot$ are binary function symbols, interpreted as the addition and multiplication of natural numbers,

- the binary relation $=$, which denotes equality of two natural numbers.

Find formulas of predicate logic for the following statements:

(a) x is a prime number (use a free variable x).

(b) z is the greatest common divisor of x and y (use free variables x, y, z).

(c) x and y are coprime (use free variables x and y).

(d) There is no largest prime number.

(e) Every number except for 1 is the product of a prime number and a natural number.

(f) Every prime number except for 2 is odd.

(g) Every even number which is greater than 2 is a sum of two prime numbers (Goldbach's conjecture).

(h) There are infinitely many prime numbers p, such that $p+2$ is a prime number as well.

Solution:

(a) First, we define $x = 1$ for a variable x as $\forall y(x \cdot y = y)$. Then

$$\text{prim}(x) := \neg(x = 1) \wedge \forall u \forall v\, ((u \cdot v = x) \rightarrow ((u = 1) \vee (v = 1)))$$

(b) We define $x \leq y$ as $\exists z(x + z = y)$. Furthermore, we define $z|x, y$ (z divides x and y) as $\exists u \exists v((x = u \cdot z) \wedge (y = v \cdot z))$. Then

$$z = \text{gcd}(x, y) := (z|x, y) \wedge \forall u((u|x, y) \rightarrow (u \leq z)).$$

(c) $(z = 1) \wedge (z = \text{gcd}(x, y))$

(d) First, we define $x < y$ as $(x \leq y) \wedge \neg(x = y)$. Then

$$\forall x(\text{prim}(x) \rightarrow \exists y(\text{prim}(y) \wedge (x < y))).$$

(e) $\forall x(\neg(x = 1) \rightarrow \exists y \exists z(\text{prim}(y) \wedge (x = y \cdot z)))$

(f) We define $\text{odd}(x)$ (x is odd) as $\neg\exists y(x = y + y)$. Furthermore, we define $x = 2$ as $\exists y((y = 1) \wedge (x = y + y))$. Then

$$\forall x(\neg(x = 2) \rightarrow (\text{prim}(x) \rightarrow \text{odd}(x)))$$

(g) We define $\text{even}(x) = \neg\text{odd}(x)$ (x is even). Then

$$\forall x((\text{even}(x) \wedge \exists y((y = 2) \wedge (y < x))) \rightarrow \exists p \exists q(\text{prim}(p) \wedge \text{prim}(q) \wedge (x = p + q)))$$

(h) First, we define $x = y + 2$ as $\exists w(w = 2 \wedge x = y + w)$. Then

$$\forall x \exists y(\text{prim}(y) \wedge (x < y) \wedge \exists z(\text{prim}(z) \wedge (z = y + 2))))$$