

Exercise 3

Task 1

Let \mathcal{A} be a structure. Show that $\text{Th}(\mathcal{A})$ is decidable if and only if $\text{Th}(\mathcal{A}_{rel})$ is decidable.

Solution:

We have to show two directions:

The direction „If $\text{Th}(\mathcal{A}_{rel})$ is decidable then $\text{Th}(\mathcal{A})$ is decidable“ was already shown in the lecture (slides 32-33).

It remains to show the other direction: „If $\text{Th}(\mathcal{A})$ is decidable then $\text{Th}(\mathcal{A}_{rel})$ is decidable“. For this, we transform a formula F (such that \mathcal{A}_{rel} is suitable for F) into a formula F' (such that \mathcal{A} is suitable for F'), such that $\mathcal{A}_{rel} \models F$ holds if and only if $\mathcal{A} \models F'$. Under the assumption that $\text{Th}(\mathcal{A})$ is decidable, we then obtain the following decision algorithm for $\text{Th}(\mathcal{A}_{rel})$ which checks whether $F \in \text{Th}(\mathcal{A}_{rel})$:

- (i) Construct the formula F' from F , such that $\mathcal{A}_{rel} \models F$ holds if and only if $\mathcal{A} \models F'$,
- (ii) Use the decision algorithm for $\text{Th}(\mathcal{A})$ in order to check whether $F' \in \text{Th}(\mathcal{A})$,
- (iii) Output „yes“, if $F' \in \text{Th}(\mathcal{A})$ and „no“ otherwise.

Let F be a formula, such that \mathcal{A}_{rel} is suitable for F . Every atomic formula in F is of the form $R(x_1, \dots, x_n)$, where x_1, \dots, x_n are variables (as F does not contain any function symbols). If R corresponds to a predicate symbol in \mathcal{A} , then we leave $R(x_1, \dots, x_n)$ in F' as it is. If R corresponds to a function symbol f in \mathcal{A} , then we replace $R(x_1, \dots, x_n)$ in F' by $x_n = f(x_1, \dots, x_{n-1})$. By definition of \mathcal{A}_{rel} , we find that $\mathcal{A}_{rel} \models F$ holds if and only if $\mathcal{A} \models F'$.

Task 2

Let $(\mathbb{N}, +, f)$ be a structure, where

- \mathbb{N} denotes the universe of the structure,
- $+$ denotes a binary function symbol interpreted as the addition of natural numbers, and
- f denotes a unary function symbol interpreted as the function $f : \mathbb{N} \rightarrow \mathbb{N}$ with $f(x) = x^2$.

Show that $\text{Th}(\mathbb{N}, +, f)$ is undecidable.

Solution:

Let F be a formula such that $(\mathbb{N}, +, \cdot)$ is suitable for F . We transform F into a formula F' such that $(\mathbb{N}, +, f)$ is suitable for F' , and such that $F \in \text{Th}(\mathbb{N}, +, \cdot)$ if and only if $F' \in \text{Th}(\mathbb{N}, +, f)$. Let $\mathcal{A}_{rel} = (\mathbb{N}, P_1, P_2)$ denote the relational structure corresponding to $(\mathbb{N}, +, \cdot)$ with

$$P_1 = \{(a_1, a_2, a) \in \mathbb{N}^3 \mid a_1 + a_2 = a\} \quad \text{and} \\ P_2 = \{(a_1, a_2, a) \in \mathbb{N}^3 \mid a_1 \cdot a_2 = a\}.$$

Using the construction from the lecture (slide 33), we transform F into a formula F'' , such that \mathcal{A}_{rel} is suitable for F'' and $F \in \text{Th}(\mathbb{N}, +, \cdot)$ if and only if $F'' \in \text{Th}(\mathcal{A}_{rel})$. Every atomic formula in F'' is of the form $P_1(x_1, x_2, x_3)$, $P_2(x_1, x_2, x_3)$ (or $x_1 = x_2$), where x_1, x_2, x_3 are variables. We find that $z = x \cdot y$ can be defined in $(\mathbb{N}, +, f)$ as

$$z = x \cdot y \quad := \quad f(x) + f(y) + z + z = f(x + y).$$

In order to obtain F' , replace every occurrence of $P_1(x_1, x_2, x)$ in F'' by $x_1 + x_2 = x$ and every occurrence of $P_2(x_1, x_2, x)$ in F'' by $f(x_1) + f(x_2) + x + x = f(x_1 + x_2)$: This yields a formula F' , such that $(\mathbb{N}, +, f)$ is suitable for F' and such that $F \in \text{Th}(\mathbb{N}, +, \cdot)$ if and only if $F' \in \text{Th}(\mathbb{N}, +, f)$. If $\text{Th}(\mathbb{N}, +, f)$ were decidable, we could obtain a decision algorithm for $\text{Th}(\mathbb{N}, +, \cdot)$ by transforming a formula F (for which $(\mathbb{N}, +, \cdot)$ is suitable) into a formula F' (for which $(\mathbb{N}, +, f)$ is suitable), such that $F \in \text{Th}(\mathbb{N}, +, \cdot)$ if and only if $F' \in \text{Th}(\mathbb{N}, +, f)$, and then use the decision algorithm for $\text{Th}(\mathbb{N}, +, f)$ to check whether $F' \in \text{Th}(\mathbb{N}, +, f)$. As $\text{Th}(\mathbb{N}, +, \cdot)$ is undecidable by Gödel's Theorem, $\text{Th}(\mathbb{N}, +, f)$ is undecidable as well.

Task 3

Let $(\mathbb{Z}, +, \cdot)$ be a structure, where

- \mathbb{Z} denotes the universe of the structure,
- $+$ denotes a binary function symbol interpreted as the addition of integers, and
- \cdot denotes a binary function symbol interpreted as the multiplication of integers.

Show that $\text{Th}(\mathbb{Z}, +, \cdot)$ is undecidable.

Hint: Apply Lagrange's four-square theorem:

Theorem 1 (Lagrange's four-square theorem)

Every natural number can be represented as the sum of four integer squares, that is, for every $x \in \mathbb{N}$, there are integers $x_1, x_2, x_3, x_4 \in \mathbb{Z}$, such that $x = x_1^2 + x_2^2 + x_3^2 + x_4^2$.

Solution:

Note that Lagrange's four-square theorem does not hold for negative integers. Let F be a formula such that $(\mathbb{N}, +, \cdot)$ is suitable for F . We transform F into a formula F' , such that

$(\mathbb{Z}, +, \cdot)$ is suitable for F' , and such that $F \in \text{Th}(\mathbb{N}, +, \cdot)$ if and only if $F' \in \text{Th}(\mathbb{Z}, +, \cdot)$. We define

$$\text{natural}(x) := \exists y_1 \exists y_2 \exists y_3 \exists y_4 (x = y_1 \cdot y_1 + y_2 \cdot y_2 + y_3 \cdot y_3 + y_4 \cdot y_4).$$

In order to obtain F' , we replace every occurrence of a subformula $\exists x_i G$ by

$$\exists x_i (G \wedge \text{natural}(x_i))$$

and every occurrence of a subformula $\forall x_i G$ by

$$\forall x_i (\text{natural}(x_i) \rightarrow G).$$

This yields a formula F' , such that $(\mathbb{Z}, +, \cdot)$ is suitable for F' and such that $F \in \text{Th}(\mathbb{N}, +, \cdot)$ if and only if $F' \in \text{Th}(\mathbb{Z}, +, \cdot)$. By Gödel's Theorem, $\text{Th}(\mathbb{N}, +, \cdot)$ is undecidable. Thus, $\text{Th}(\mathbb{Z}, +, \cdot)$ is undecidable as well.