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Exercise 4

Task 1
Consider the structure $(\mathbb{N}, +, \cdot, s, 0)$. Use Gödel's β-function in order to formalize the following statements in predicate logic:

(a) $x^y = z$ (use free variables x, y and z),

(b) Fermat's Last Theorem,

(c) Collatz conjecture.

Task 2
Show that the set of valid formulas of predicate logic is undecidable. Use a reduction to the Post correspondence problem for the proof.

Post correspondence problem (PCP)
Input: A sequence of pairs $(s_1, t_1), \dots, (s_k, t_k)$, such that $k \geq 0, s_i, t_i \in \{0,1\}^*$ $(1 \leq i \leq k)$.
Question: Are there indices $i_1, \dots, i_m \in \{1, \dots, k\}$ $(m \geq 1)$ such that $s_{i_1} \cdots s_{i_m} = t_{i_1} \cdots t_{i_m}$?

Hint: Let e be a 0-ary function symbol, f_0 and f_1 be unary function symbols and P denote a binary predicate symbol. Let $z = c_1 \dots c_\ell \in \{0,1\}^\ell$ $(\ell \geq 0)$ be a string and let t be a term. We define $f_z(t)$ as $f_{c_\ell}(\dots(f_{c_1}(t))\dots)$. Let

$$\phi_1 = \bigwedge_{i=1}^{k} P(f_{s_i}(e), f_{t_i}(e)),$$

$$\phi_2 = \forall v \forall w (P(v, w) \rightarrow \bigwedge_{i=1}^{k} P(f_{s_i}(v), f_{t_i}(w))),$$

$$\phi_3 = \exists z (P(z, z))$$

and let $\phi = \phi_1 \wedge \phi_2 \rightarrow \phi_3$. Show that ϕ is valid if and only if the corresponding instance $(s_1, t_1), \dots, (s_k, t_k)$ of the post correspondence problem has a solution.