Universität Siegen
Lehrstuhl Theoretische Informatik
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Logic II
SS 2021

Exercise 6

Task 1
Which of the following statements are correct? Give reasons for your answer.

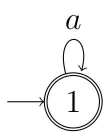
(a) $(\mathbb{N}, \leq)$ is automatically presentable.

(b) Let $M \subseteq \mathbb{N}$ (unary relation), then $(\mathbb{N}, M)$ is automatically presentable.

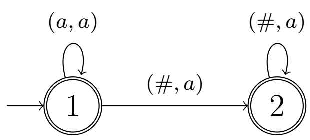
(c) If $(\mathbb{N}, R_1)$ and $(\mathbb{N}, R_2)$ are automatically presentable, then $(\mathbb{N}, R_1, R_2)$ is automatically presentable.

Solution:

(a) This statement is correct: Let $f \colon \mathbb{N} \to \{a\}^*$ be defined by $f(i) = a^i$. Let $(\{a\}^*, \leq_a)$ with $a^i \leq_a a^j$ if and only if $i \leq j$. Then $(\mathbb{N}, \leq)$ and $(\{a\}^*, \leq_a)$ are isomorphic and f is the corresponding isomorphism, as $i \leq j$ if and only if $f(i) = a^i \leq a^j = f(j)$. Furthermore, f is bijective. Moreover, $(\{a\}^*, \leq_a)$ is automatic, as



is a finite automaton for $\{a\}^*$ and



is a 2-tape automaton for $\leq_a$.

(b) This statement is correct: If M or $\mathbb{N} \setminus M$ is finite, then let $(\{a\}^*, P)$ with $P = \{a^i \mid i \in M\}$ and $f(i) = a^i$. We find that $(\{a\}^*, P)$ and $(\mathbb{N}, M)$ are isomorphic and f is the corresponding isomorphism. The automaton that accepts $\{a\}^*$ is shown in part (a). If M is finite, then P is finite and thus accepted by a finite automaton as finite languages are always regular (recall that P is a unary relation and a 1-tape automaton is a „standard“ finite automaton).

If $\mathbb{N} \setminus M$ is finite, then the complement of P is finite and hence regular. As regular languages are closed under taking the complement, we find that P is regular and thus there is a finite automaton which accepts P. Thus, $(\{a\}^*, P)$ is automatic in this case.

If both M and $\mathbb{N} \setminus M$ are infinite, then let $M = \{a_0, a_1, a_2, \dots\}$ and let $\mathbb{N} \setminus M = \{b_1, b_2, \dots\}$ (note that both M and $\mathbb{N} \setminus M$ are countable as subsets of $\mathbb{N}$). We define $(\{a\}^* \cup \{b\}^*, P)$ by $P = \{a\}^*$ and $f \colon \mathbb{N} \to \{a\}^* \cup \{b\}^*$ by

$$f(i) = \begin{cases} a^j & \text{if } i \in M, a_j = i, \\ b^j & \text{if } i \notin M, b_j = i. \end{cases}$$

Then f is an isomorphism, as f is bijective and $f(i) \in P$ holds if and only if $i \in M$. Furthermore, we find that $(\{a\}^* \cup \{b\}^*, P)$ is automatic, as

is an automaton for $\{a\}^* \cup \{b\}^*$ and $P = \{a\}^*$ is accepted by the finite automaton in part (a).

(c) The statement is not correct: By parts (a) and (b), we know that $(\mathbb{N}, \leq)$ and $(\mathbb{N}, M)$ are automatically presentable, where $M \subset \mathbb{N}$ is a unary relation. However, we show that $(\mathbb{N}, \leq, M)$ is not necessarily automatically presentable: We can define every natural number $n \in \mathbb{N}$ using $\leq$ (and $=$): Let a and b be free variables. Define $a < b$ by $a \leq b \wedge \neg(a = b)$. We define the following formulas:

$$\begin{aligned} s(a, b) &= \neg\exists z(a < z \wedge z < b), \\ 0(a) &= \neg\exists z\ z < a. \end{aligned}$$

The formula $s(a, b)$ states that there is no natural number which is greater than a and smaller than b (that is, b is the immediate successor of a). The formula $0(a)$ defines the natural number 0, as there is no natural number, which is smaller than 0. Furthermore, we define $s^0(a) = 0(a)$ and for every $i \in \mathbb{N}$ let

$$s^{i+1}(a) = \exists x_i(s^i(x_i) \wedge s(x_i, a)).$$

Then $s^n(a)$ states that the free variable a is the natural number n. Let M be an undecidable subset of $\mathbb{N}$. If the structure $(\mathbb{N}, \leq, M)$ were automatically presentable, then $\mathrm{Th}(\mathbb{N}, \leq, M)$ would be decidable by the Theorem of Khoussainov/Nerode. Then we could check if $n \in M$, by checking if $\forall x(s^n(x) \to M(x)) \in \mathrm{Th}(\mathbb{N}, \leq, M)$. As M is undecidable, we obtain a contradiction. Thus, $(\mathbb{N}, \leq, M)$ cannot be automatically presentable.

Task 2
Are any two countable linear orders without a smallest and a largest element isomorphic?

Solution:
We find that $(\mathbb{Z}, \leq)$ and $(\mathbb{Q}, \leq)$ are countable linear orders without a smallest and a largest element, but they are not isomorphic: For example, we find that $(\mathbb{Q}, \leq)$ is dense, but $(\mathbb{Z}, \leq)$ is not dense. In order to show a contradiction, assume that there is a bijection $h : \mathbb{Z} \to \mathbb{Q}$, such that

$$a \leq b \quad \Longleftrightarrow \quad h(a) \leq h(b)$$

holds for all $a, b \in \mathbb{Z}$. Fix two elements $a, b \in \mathbb{Z}$ such that $a + 1 = b$. As $\mathbb{Q}$ is dense, there is an element $q \in \mathbb{Q}$, such that $h(a) < q < h(b)$. As h is a bijection, we have $q = h(c)$ for an element $c \in \mathbb{Z}$. However, we either have $c < a$ or $b < c$, as $a + 1 = b$. This yields a contradiction.