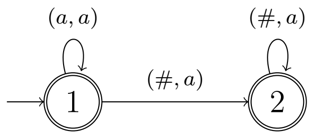
Exercise 7

Task 1

Check whether $(\mathbb{N}, \leq) \models \exists x \forall y (x \leq y)$ holds by applying the technique from the proof of the Theorem of Khoussainov and Nerode.

Solution:

In the solution of exercise sheet 6, task 1, we showed that $(\mathbb{N}, \leq)$ is isomorphic to the following automatic structure: $(\{a\}^*, \leq_a)$, where $a^i \leq_a a^j$ if and only if $i \leq j$. In particular,



is a 2-tape automaton for $\leq_a$.

As a first step, we transform the formula $\exists x \forall y (x \leq_a y)$ into an equivalent formula which does not contain a $\forall$-quantifier: A $\forall$-quantifier can be expressed using a negation of an $\exists$-quantifier. We find

$$\exists x \forall y (x \leq_a y) \equiv \exists x \neg \exists y \neg (x \leq_a y).$$

Now let $F = \exists x \neg \exists y \neg (x \leq_a y)$. We start with the (atomic) subformula $F_1 = x \leq_a y$, which is treated in case 1 (slide 75) in the proof of the Theorem of Khoussainov und Nerode. This means that we construct a synchronous 2-tape automaton B_{F_1}, such that

$$K(B_{F_1}) = \{(w_1, w_2) \in \{a\}^* \times \{a\}^* \mid w_1 \leq_a w_2\}.$$

In this concrete example, we can take the 2-tape automaton from above, which accepts precisely this relation. Note that all variables of F are free variables in F_1, and we assume that they are ordered according to their occurrence in F_1 – thus, the homomorphism from case 1 on slide 75 is the identity mapping.

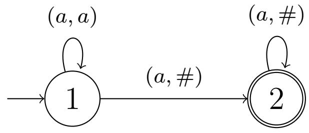
Next, we consider the subformula $F_2 = \neg F_1 = \neg(x \leq_a y)$: This corresponds to case 3 (slide 77) from the proof of the Theorem of Khoussainov and Nerode. We thus need a 2-tape automaton B_{F_2}, such that

$$L(B_{F_2}) = \{w_1 \otimes w_2 \mid w_1, w_2 \in \{a\}^*\} \setminus L(B_{F_1}),$$

respectively,

$$K(B_{F_2}) = \{(a^n, a^m) \mid n > m\}.$$

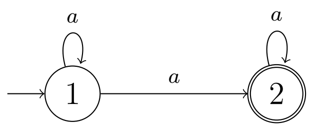
The following automaton satisfies this property:



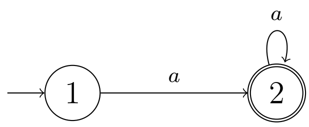
Next, we consider the subformula $F_3 = \exists y F_2 = \exists y \neg(x \leq_a y)$. This corresponds to case 5 (slide 78). Let f be the homomorphism defined by $f(w_1 \otimes w_2) = w_1$ (slide 78). We are looking for an automaton B_{F_3}, such that

$$L(B_{F_3}) = f(L(B_{F_2})).$$

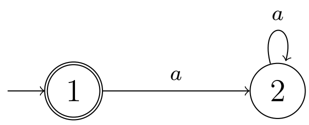
This means that we simply ignore the second component:



This non-deterministic automaton accepts the same language as the following deterministic automaton:



Next, we consider the subformula $F_4 = \neg F_3 = \neg\exists y \neg(x \leq_a y)$, which again corresponds to case 3 (slide 77). It is easy to see that the complement of $L(B_{F_3})$ only contains the empty word ε. Hence, B_{F_4} is the following automaton (which is obtained by switching accept and non-accept states in the above deterministic automaton):



The formula F is of the form $F = \exists x F_4$ (as on slide 80). We have $(\{a\}^*, \leq_a) \models F$ if and only if $L(B_{F_4}) \neq \emptyset$. As $L(B_{F_4}) = \{\varepsilon\}$, we find that $F \in \mathrm{Th}(\mathbb{N}, \leq)$.

Task 2
Show that $\mathrm{Th}(\mathbb{C}, +, \cdot)$ is decidable.

Solution:
By Tarski's Theorem, we know that $\mathrm{Th}(\mathbb{R}, +, \cdot)$ is decidable: Let F be a formula, such that $(\mathbb{C}, +, \cdot)$ is suitable for F. We transform F into a formula F', such that $(\mathbb{R}, +, \cdot)$ is suitable for F' and such that $F \in \mathrm{Th}(\mathbb{C}, +, \cdot)$ if and only if $F' \in \mathrm{Th}(\mathbb{R}, +, \cdot)$.
The main idea is that each complex number is uniquely representable by two real numbers, its real part and its imaginary part. Thus, we replace each variable x in F by two new variables x_1, x_2 in F', such that x_1 represents the real part of x and x_2 represents the imaginary part of x. Furthermore, we transform subformulas of the form $\exists x G$ into $\exists x_1 \exists x_2 G$ and subformulas of the form $\forall x G$ into $\forall x_1 \forall x_2 G$. It remains to transform $x + y = z$ into

$$(x_1 + y_1 = z_1) \wedge (x_2 + y_2 = z_2)$$

and $x \cdot y = z$ into

$$(x_1 y_1 - x_2 y_2 = z_1) \wedge (x_2 y_1 + x_1 y_2 = z_2).$$

(as $\mathrm{Th}((\mathbb{C}, +, \cdot)_{rel})$ is decidable if and only if $\mathrm{Th}(\mathbb{C}, +, \cdot)$ is decidable).