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Exercise 8

Task 1
Show that $\mathrm{Th}(\mathbb{R}, +, \cdot)$ is decidable if and only if $\mathrm{Th}(\mathbb{R}, +, \cdot, <, 0, 1, -1)$ is decidable.

Solution:
We have to show two directions:
If $\mathrm{Th}(\mathbb{R}, +, \cdot, <, 0, 1, -1)$ is decidable, then clearly $\mathrm{Th}(\mathbb{R}, +, \cdot)$ is decidable as well: Let F be a formula such that $(\mathbb{R}, +, \cdot)$ is suitable for F, then $(\mathbb{R}, +, \cdot, <, 0, 1, -1)$ is suitable for F as well and $F \in \mathrm{Th}(\mathbb{R}, +, \cdot)$ if and only if $F \in \mathrm{Th}(\mathbb{R}, +, \cdot, <, 0, 1, -1)$.
For the other direction, assume that $\mathrm{Th}(\mathbb{R}, +, \cdot)$ is decidable. Let F be a formula, such that $(\mathbb{R}, +, \cdot, <, 0, 1, -1)$ is suitable for F. We transform F into a formula F', such that $(\mathbb{R}, +, \cdot)$ is suitable for F' and such that $F \in \mathrm{Th}(\mathbb{R}, +, \cdot, <, 0, 1, -1)$ if and only if $F' \in \mathrm{Th}(\mathbb{R}, +, \cdot)$. We find that we can represent $x < y$ by

$$\exists z(x + z \cdot z = y) \wedge \neg(x = y).$$

Furthermore, we obtain the constant 0 as

$$\exists x_0 \forall y(y + x_0 = y)$$

and the constants 1 and -1 as

$$\exists x_1 \exists x_{-1} \forall y(y \cdot x_1 = y) \wedge (x_{-1} \cdot x_{-1} = x_1) \wedge \neg(x_1 = x_{-1}).$$

Task 2
Show that the set of natural numbers $\mathbb{N}$ cannot be defined in $(\mathbb{R}, +, \cdot)$ (without using the fact that $\mathrm{Th}(\mathbb{N}, +, \cdot)$ is undecidable).

Solution:
Assume that there is a formula F with one free variable x which defines $\mathbb{N}$ in $(\mathbb{R}, +, \cdot, <, 0, 1, -1)$, that is, $(\mathbb{R}, +, \cdot, <, 0, 1, -1) \models F[n/x]$ if and only if $n \in \mathbb{N}$. Using quantifier elimination, we can transform F into and equivalent formula F' without quantifiers. We assume that F' is of the form

$$(s(x) = 0) \wedge \bigwedge_{i=1}^{m} (t_i(x) > 0),$$

where $s, t_1, \ldots, t_m \in \mathbb{Z}[x]$ are polynomials (see slide 88 of the lecture). Let $d(p)$ be the degree of a polynomial p. We find that s is either the zero polynomial, or that there are at most $d(s)$ many solutions of the equation $s(x) = 0$. The set of all real numbers x which

satisfy $s(x) = 0$ is thus either an interval (which is equal to $\mathbb{R}$) or the union of $d(s)$ many singleton sets.
Furthermore, we find for every index i ($1 \leq i \leq m$), that the set of real numbers x which satisfy $t_i(x) > 0$ is a union of at most $d(t_i)$ many intervals. We can represent boolean combinations (e.g. intersections) of finitely many invervals over $\mathbb{R}$ using finitely many intervals over $\mathbb{R}$. Thus, if F characterizes the natural numbers, then $\mathbb{N}$ must be representable by finitely many intervals over $\mathbb{R}$ – this is a contradiction.