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Exercise 9

Task 1

Let $f, g \in \mathbb{R}[x]$ with

$$f(x) = x^4 + x^3 - x - 1,$$
$$g(x) = 4x^3 + 3x^2 - 1.$$

Compute the Sturm sequence $[f, g]$.

Solution:

We apply Euclid's algorithm for f and g.

1. Step:

$$\begin{array}{l} \big(x^4 + x^3 - x - 1\big) / \big(4x^3 + 3x^2 - 1\big) = \frac{1}{4}x + \frac{1}{16} + \dfrac{-\frac{3}{16}x^2 - \frac{3}{4}x - \frac{15}{16}}{4x^3 + 3x^2 - 1} \\ -x^4 - \frac{3}{4}x^3 + \frac{1}{4}x \\ \hline \frac{1}{4}x^3 - \frac{3}{4}x - 1 \\ -\frac{1}{4}x^3 - \frac{3}{16}x^2 + \frac{1}{16} \\ \hline -\frac{3}{16}x^2 - \frac{3}{4}x - \frac{15}{16} \end{array}$$

2. Step:

$$\begin{array}{l} \big(4x^3 + 3x^2 - 1\big) / \big(\frac{3}{16}x^2 + \frac{3}{4}x + \frac{15}{16}\big) = \frac{64}{3}x - \frac{208}{3} + \dfrac{32x + 64}{\frac{3}{16}x^2 + \frac{3}{4}x + \frac{15}{16}} \\ -4x^3 - 16x^2 - 20x \\ \hline -13x^2 - 20x - 1 \\ 13x^2 + 52x + 65 \\ \hline 32x + 64 \end{array}$$

3. Step:

$$\begin{array}{l} \big(\frac{3}{16}x^2 + \frac{3}{4}x + \frac{15}{16}\big) / \big(-32x - 64\big) = -\frac{3}{512}x - \frac{3}{256} + \dfrac{\frac{3}{16}}{-32x - 64} \\ -\frac{3}{16}x^2 - \frac{3}{8}x \\ \hline \frac{3}{8}x + \frac{15}{16} \\ -\frac{3}{8}x - \frac{3}{4} \\ \hline \frac{3}{16} \end{array}$$

4. Step:

$$\begin{array}{l} \big(-32x - 64\big) \,/\, -\tfrac{3}{16} = \tfrac{512}{3}x + \tfrac{1024}{3} \\ \underline{\quad 32x \quad} \\ \qquad\quad -64 \\ \qquad\quad \underline{\ \ 64\ \ } \\ \qquad\qquad 0 \end{array}$$

The Sturm sequence $[f,g]$ consists of f, g and the negative remainders of Euclid's algorithm for f and g:

$$[f,g] = \left(f, g, \frac{3}{16}x^2 + \frac{3}{4}x + \frac{15}{16}, -32x - 64, -\frac{3}{16}\right)$$

Task 2
Compute the number of zeros of $f(x) = x^4 - x^2 - 1$ in the interval $(-2, 2)$ using the Theorem of Sturm and Tarski.

Solution:
Consider f and the derivative f':

$$\begin{aligned} f(x) &= x^4 - x^2 - 1 \\ f'(x) &= 4x^3 - 2x \end{aligned}$$

We apply Euclid's algorithm for f and f':

1. Step:

$$\begin{array}{l} \big(\ \ x^4 \ \ - x^2 - 1\big) \,/\, \big(4x^3 - 2x\big) = \tfrac{1}{4}x + \dfrac{-\tfrac{1}{2}x^2 - 1}{4x^3 - 2x} \\ \underline{\ -x^4 + \tfrac{1}{2}x^2\ } \\ \qquad\ \ -\tfrac{1}{2}x^2 \end{array}$$

2. Step:

$$\begin{array}{l} \big(\ \ 4x^3 \ \ - 2x\big) \,/\, \big(\tfrac{1}{2}x^2 + 1\big) = 8x + \dfrac{-10x}{\tfrac{1}{2}x^2 + 1} \\ \underline{\ -4x^3 \ \ - 8x\ } \\ \qquad\quad -10x \end{array}$$

3. Step:

$$\begin{array}{l} \big(\ \ \tfrac{1}{2}x^2 + 1\big) \,/ 10x = \tfrac{1}{20}x + \dfrac{1}{10x} \\ \underline{\ -\tfrac{1}{2}x^2\ } \end{array}$$

4. Step:

$$\begin{array}{l} \big(-10x\big) \,/ -1 = 10x \\ \underline{\quad 10x\quad} \\ \qquad\ \ 0 \end{array}$$

Sturm sequence:

$$[f, f'] = \left(x^4 - x^2 - 1, 4x^3 - 2x, \frac{1}{2}x^2 + 1, 10x, -1 \right)$$

With $a = -2$:

$$(11, -28, 3, -20, -1)$$

With $b = 2$:

$$(11, 28, 3, 20, -1)$$

There are 3 changes of sign for a and one change of sign for b, thus we get $3 - 1 = 2$ zeros of f in the interval $(-2, 2)$.