Algorithms I

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Wintersemester 2018/2019

Overview, Literature

Overview:

1. Basics
2. Divide & Conquer
3. Sorting
4. Greedy algorithms
5. Dynamic programming
6. Graph algorithms

Literature:

- Cormen, Leiserson Rivest, Stein. Introduction to Algorithms (3. Auflage); MIT Press 2009
- Schöning, Algorithmik. Spektrum Akademischer Verlag 2001

Landau Symbols

Let $f, g : \mathbb{N} \to \mathbb{N}$ be functions.

- $g \in \mathcal{O}(f)$ if
  $$\exists c > 0 \, \exists n_0 \, \forall n \geq n_0 : \; g(n) \leq c \cdot f(n).$$
  In other words: g is not growing faster than f.
- $g \in o(f)$ if
  $$\forall c > 0 \, \exists n_0 \, \forall n \geq n_0 : \; g(n) \leq c \cdot f(n).$$
  In other words: g is growing strictly slower than f.
- $g \in \Omega(f) \;\Leftrightarrow\; f \in \mathcal{O}(g)$
  In other words: g is growing at least as fast than f.
- $g \in \omega(f) \;\Leftrightarrow\; f \in o(g)$
  In other words: g is growing strictly faster than f.
- $g \in \Theta(f) \;\Leftrightarrow\; (f \in \mathcal{O}(g) \wedge g \in \mathcal{O}(f))$
  In other words: g and f have the same asymptotic growth.

Complexity measures

We describe the running time of an algorithm A as a function in the input length n.

Standard: Worst case complexity

Maximal running time on all inputs of length n:

$$t_{A,\text{worst}}(n) = \max\{t_A(x) \mid x \in X_n\},$$

where $X_n = \{x \mid |x| = n\}$.

Criticism: Unrealistic, since in practise worst-case inputs might not arise.

Complexity measures

Alternative: average case complexity.

Needs a probability distribution on X_n.

Standard: uniform distribution, i.e., $\mathrm{Prob}(x) = \frac{1}{|X_n|}$.

Average running time:

$$
\begin{aligned}
t_{A,\varnothing}(n) &= \sum_{x \in X_n} \mathrm{Prob}(x) \cdot t_A(x) \\
&= \frac{1}{|X_n|} \sum_{x \in X_n} t_A(x) \qquad \text{(for uniform distribution)}
\end{aligned}
$$

Problem: Difficult to analyse

Example: quicksort

Worst case number of comparisons of quicksort: $t_Q(n) \in \Theta(n^2)$.

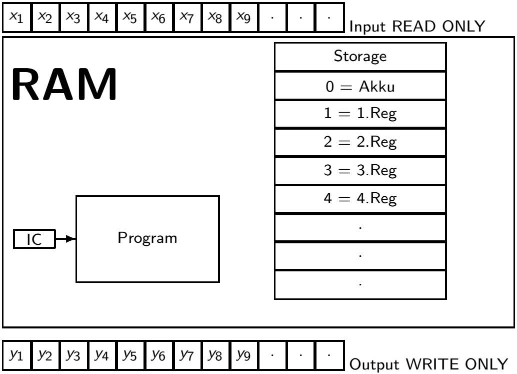
Average number of comparisons: $t_{Q,\varnothing}(n) = 1.38 n \log n$

Machine models: Turing machines

The Turing machine (TM) is a very simple and mathematically easy to define model of computation.

But: memory access (i.e., moving head to a certain symbol on the tape) is very time-consuming on a Turing machine and not realistic.

Machine models: Register machine (RAM)



Assumption: Elementary operations (e.g., the arithmetic operations $+, \times, -$, DIV, comparison, bitwise AND and OR) need a single computation step.

Overview

- Solving recursive equations
- Mergesort
- Fast multiplication of integers
- Matrix multiplication a la Strassen

Divide & Conquer: basic idea

As a first major design principle for algorithms, we will see Divide & Conquer:

Basic idea:

- Divide the input into several parts (usually of roughly equal size)
- Solve the problem on each part separately (recursion).
- Construct the overall solution from the sub-solutions.

Recursive equations

Divide & Conquer leads in a very natural way to recursive equations.

Assumptions:

- Input of length n will be split into a many parts of size n/b.
- Dividing the input and merging the sub-solutions takes time $g(n)$.
- For an input of length 1 the computation time is $g(1)$.

This leads to the following recursive equation for the computation time:

$$\begin{aligned} t(1) &= g(1) \\ t(n) &= a \cdot t(n/b) + g(n) \end{aligned}$$

Technical probem: What happens, if n is not divisible by b?

- Solution 1: Replace n/b by $\lceil n/b \rceil$.
- Solution 2: Assume that $n = b^k$ for some $k \geq 0$.

  If this does not hold: Stretch the input (for every n there exists a $k \geq 0$ with $n \leq b^k < bn$).

Solving simple recursive equations

Theorem 1

Let $a, b \in \mathbb{N}$ *and* $b > 1$, $g : \mathbb{N} \longrightarrow \mathbb{N}$ *and assume the following equations:*

$$\begin{aligned} t(1) &= g(1) \\ t(n) &= a \cdot t(n/b) + g(n) \end{aligned}$$

Then for all $n = b^k$ *(i.e.,* $k = \log_b(n)$*):*

$$t(n) = \sum_{i=0}^{k} a^i \cdot g\left(\frac{n}{b^i}\right).$$

Proof: Induction over k.

$k = 0$: We have $n = b^0 = 1$ and $t(1) = g(1)$.

Solving simple recursive equations

$k > 0$: By induction we have

$$t\left(\frac{n}{b}\right) = \sum_{i=0}^{k-1} a^i \cdot g\left(\frac{n}{b^{i+1}}\right).$$

Hence:

$$\begin{aligned}
t(n) &= a \cdot t\left(\frac{n}{b}\right) + g(n) \\
&= a\left(\sum_{i=0}^{k-1} a^i \cdot g\left(\frac{n}{b^{i+1}}\right)\right) + g(n) \\
&= \sum_{i=1}^{k} a^i \cdot g\left(\frac{n}{b^i}\right) + a^0 g\left(\frac{n}{b^0}\right) \\
&= \sum_{i=0}^{k} a^i \cdot g\left(\frac{n}{b^i}\right).
\end{aligned}$$

$\square$

Master theorem I

Theorem 2 (Master theorem I)

Let $a, b, c, d \in \mathbb{N}$ *with* $b > 1$ *and assume that*

$$\begin{aligned} t(1) &= d \\ t(n) &= a \cdot t(n/b) + d \cdot n^c \end{aligned}$$

Then, for all n *of the form* b^k *with* $k \geq 0$ *we have:*

$$t(n) \in \begin{cases} \Theta(n^c) & \text{if } a < b^c \\ \Theta(n^c \log n) & \text{if } a = b^c \\ \Theta(n^{\frac{\log a}{\log b}}) & \text{if } a > b^c \end{cases}$$

Remark: $\frac{\log a}{\log b} = \log_b a$. If $a > b^c$, then $\log_b a > c$.

Proof of the master theorem I

Let $g(n) = dn^c$. By Theorem 1 we have the following for $k = \log_b n$:

$$t(n) = d \cdot n^c \cdot \sum_{i=0}^{k} \left(\frac{a}{b^c}\right)^i.$$

Case 1: $a < b^c$

$$t(n) \le d \cdot n^c \cdot \sum_{i=0}^{\infty} \left(\frac{a}{b^c}\right)^i = d \cdot n^c \cdot \frac{1}{1 - \frac{a}{b^c}} \in \mathcal{O}(n^c).$$

Moreover, $t(n) \in \Omega(n^c)$, which implies $t(n) \in \Theta(n^c)$.

Case 2: $a = b^c$

$$t(n) = (k+1) \cdot d \cdot n^c \in \Theta(n^c \log n).$$

Proof of the master theorem I

Case 3: $a > b^c$

$$
\begin{aligned}
t(n) &= d \cdot n^c \cdot \sum_{i=0}^{k} \left(\frac{a}{b^c}\right)^i = d \cdot n^c \cdot \frac{\left(\frac{a}{b^c}\right)^{k+1} - 1}{\frac{a}{b^c} - 1} \\
&\in \Theta\left(n^c \cdot \left(\frac{a}{b^c}\right)^{\log_b(n)}\right) \\
&= \Theta\left(\frac{n^c \cdot a^{\log_b(n)}}{b^{c \log_b(n)}}\right) \\
&= \Theta\left(a^{\log_b(n)}\right) \\
&= \Theta\left(b^{\log_b(a) \cdot \log_b(n)}\right) \\
&= \Theta\left(n^{\log_b(a)}\right)
\end{aligned}
$$

$\square$

Stretching the input is ok

Stretching the input length to a b-power does not change the statement of the master theorem I.

Formally: Assume that the function t satisfies the following recursive equation

$$\begin{aligned} t(1) &= d \\ t(n) &= a \cdot t(n/b) + d \cdot n^c \end{aligned}$$

for all b-powers n.

Define the function $s : \mathbb{N} \to \mathbb{N}$ by $s(n) = t(m)$, where m is the smallest b-power with $m \geq n$ ($\rightsquigarrow n \leq m \leq bn$).

With the master theorem I we get

$$s(n) = t(m) \in \begin{cases} \Theta(m^c) = \Theta(n^c) & \text{if } a < b^c \\ \Theta(m^c \log m) = \Theta(n^c \log n) & \text{if } a = b^c \\ \Theta(m^{\frac{\log a}{\log b}}) = \Theta(n^{\frac{\log a}{\log b}}) & \text{if } a > b^c \end{cases}$$

Mergesort

We want to sort an array A of length n, where $n = 2^k$ for some $k \geq 0$.

Algorithm mergesort

---

procedure mergesort(l, r)
var m : **integer**;
begin
  if $(l < r)$ **then**
    $m := (r + l)$ div 2;
    mergesort(l, m);
    mergesort($m + 1, r$);
    merge(l, m, r);
  endif
endprocedure

Mergesort

Algorithm merge

procedure merge(l, m, r)
var i, j, k : **integer**;
begin
  $i = l$; $j := m + 1$;
  for $k := 1$ **to** $r - l + 1$ **do**
    if $i = m + 1$ **or** ($i \leq m$ **and** $j \leq r$ **and** $A[j] \leq A[i]$) **then**
      $B[k] := A[j]$; $j := j + 1$
    else
      $B[k] := A[i]$; $i := i + 1$
    endif
  endfor
  for $k := 0$ **to** $r - l$ **do**
    $A[l + k] := B[k + 1]$
  endfor
endprocedure

Mergesort

Note: merge(l, m, r) works in time $\mathcal{O}(r - l + 1)$.

Running time: $t_{\text{ms}}(n) = 2 \cdot t_{\text{ms}}(n/2) + d \cdot n$ for a constant d.

Master theorem I: $t_{\text{ms}}(n) \in \Theta(n \log n)$.

We will see later that $\mathcal{O}(n \log n)$ is asymptotically optimal for sorting algorithms that are only based on the comparison of elements.

Drawback of Mergesort: no in-place sorting algorithm

A sorting algorithm works in-place, if at every time instant only a constant number of elements from the input array A is stored outside of A.

We will see in-place sorting algorithms with a running of $\mathcal{O}(n \log n)$.

Multiplication of natural numbers

We want to multiply two n-bit natural numbers, where $n = 2^k$ for some $k \geq 0$.

School method: $\Theta(n^2)$ bit operations.

Alternative approach:

$r =$	A	B
$s =$	C	D

Here, A (C) are the first $n/2$ bits and B (D) are the last $n/2$ bits of r (s), i.e.,

$$r = A\,2^{n/2} + B; \qquad s = C\,2^{n/2} + D$$

$$r\,s = A\,C\,2^n + (A\,D + B\,C)\,2^{n/2} + B\,D$$

Master theorem I: $t_{\text{mult}}(n) = 4 \cdot t_{\text{mult}}(n/2) + \Theta(n) \in \Theta(n^2)$
No improvement!

Fast multiplication by A. Karatsuba, 1960

Compute recursively AC, $(A - B)(D - C)$ and BD.

Then, we get

$$rs = A\,C\,2^n + (A - B)\,(D - C)\,2^{n/2} + (B\,D + A\,C)\,2^{n/2} + B\,D$$

By the master theorem I, the total number of bit operations is:

$$t_{\text{mult}}(n) = 3 \cdot t_{\text{mult}}(n/2) + \Theta(n) \in \Theta(n^{\frac{\log 3}{\log 2}}) = \Theta(n^{1.58496\ldots}).$$

Using divide & conquer we reduced the exponent from 2 (school method) to 1.58496... .

How fast can we multiply?

In 1971, Arnold Schönhage and Volker Strassen presented an algorithm which multiplies two n-bit number in time $\mathcal{O}(n \log n \log\log n)$ on a multitape Turing-machine.

The Schönhage-Strassen algorithm uses the so-called fast Fourier transformation (FFT); see Algorithms II.

In practice, the Schönhage-Strassen algorithm beats Karatsuba's algorithm for numbers with approx. 10.000 digits.

In 2007, Martin Fürer came up with an algorithm that beats the Schönhage-Strassen algorithm. His algorithm has a running time of $\mathcal{O}(n \log n\, 2^{\log^* n})$, where $\log^* n$ is the first number k such that $\log_2(\cdot)$ k-times applied to n yields a number ≤ 1.

Matrix multiplication using naive divide & conquer

Let $A = (a_{i,j})_{1 \le i,j \le n}$ and $B = (b_{i,j})_{1 \le i,j \le n}$ be two $(n \times n)$-matrices.

For the product matrix $AB = (c_{i,j})_{1 \le i,j \le n} = C$ we have

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$$

$\rightsquigarrow \Theta(n^3)$ scalar multiplications.

Divide & conquer: A, B are divided in 4 submatrices of roughly equal size. Then, the product $AB = C$ can be computed as follows:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

Matrix multiplication using naive divide-and-conquer

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where

$$\begin{aligned} C_{11} &= A_{11}B_{11} + A_{12}B_{21} \\ C_{12} &= A_{11}B_{12} + A_{12}B_{22} \\ C_{21} &= A_{21}B_{11} + A_{22}B_{21} \\ C_{22} &= A_{21}B_{12} + A_{22}B_{22} \end{aligned}$$

We get

$$t(n) = 8 \cdot t(n/2) + \Theta(n^2) \in \Theta(n^3).$$

No improvement!

Matrix multiplication by Volker Strassen (1969)

Compute the product of two 2×2 matrices with 7 multiplications:

$$
\begin{aligned}
M_1 &:= (A_{12} - A_{22})(B_{21} + B_{22}) \\
M_2 &:= (A_{11} + A_{22})(B_{11} + B_{22}) \\
M_3 &:= (A_{11} - A_{21})(B_{11} + B_{12}) \\
M_4 &:= (A_{11} + A_{12})B_{22} \\
M_5 &:= A_{11}(B_{12} - B_{22}) \\
M_6 &:= A_{22}(B_{21} - B_{11}) \\
M_7 &:= (A_{21} + A_{22})B_{11}
\end{aligned}
\qquad
\begin{aligned}
C_{11} &= M_1 + M_2 - M_4 + M_6 \\
C_{12} &= M_4 + M_5 \\
C_{21} &= M_6 + M_7 \\
C_{22} &= M_2 - M_3 + M_5 - M_7
\end{aligned}
$$

Running time: $t(n) = 7 \cdot t(n/2) + \Theta(n^2)$.

Master theorem I ($a = 7,\ b = 2,\ c = 2$):

$$t(n) \in \Theta(n^{\log_2 7}) = \Theta(n^{2,81\ldots}) \ .$$

The story of fast matrix multiplication

- Strassen 1969: $n^{2,81\ldots}$
- Pan 1979: $n^{2,796\ldots}$
- Bini, Capovani, Romani, Lotti 1979: $n^{2,78\ldots}$
- Schönhage 1981: $n^{2,522\ldots}$
- Romani 1982: $n^{2,517\ldots}$
- Coppersmith, Winograd 1981: $n^{2.496\ldots}$
- Strassen 1986: $n^{2,479\ldots}$
- Coppersmith, Winograd 1987: $n^{2.376\ldots}$
- Stothers 2010: $n^{2,374\ldots}$
- Williams 2014: $n^{2,372873\ldots}$

Overview

- Lower bounds for comparison-based sorting algorithms
- Quicksort
- Heapsort
- Sorting in linear time
- Median computation

Comparison-based sorting algorithms

A sorting algorithm is comparison-based if the elements of the input array belong to a data type that only supports the comparison of two elements.

We assume in the following considerations that the input array $A[1, \dots, n]$ has the following properties:

- $A[i] \in \{1, \dots, n\}$ for all $1 \leq i \leq n$.
- $A[i] \neq A[j]$ for $i \neq j$

In other words: The input is a permutation of the list $[1, 2, \dots, n]$.

The sorting algorithm has to sort this list.

Another point of view: The sorting algorithm has to compute the permutation $[i_1, i_2, \dots, i_n]$ such that $A[i_k] = k$ for all $1 \leq k \leq n$.

Example: On input $[2, 3, 1]$ the output should be $[3, 1, 2]$.

Lower bound for the worst case

Theorem 3

For every comparison-based sorting algorithm and every n there exists an array of length n, on which the algorithm makes at least

$$n\log_2(n) - \log_2(e)n \geq n\log_2(n) - 1,443n$$

many comparisons.

Proof: We execute the algorithm on an array $A[1,\ldots,n]$ without knowing the concrete values $A[i]$.

This yields a decision tree that can be constructed as follows:

Assume that the algorithm compares $A[i]$ and $A[j]$ in the first step.

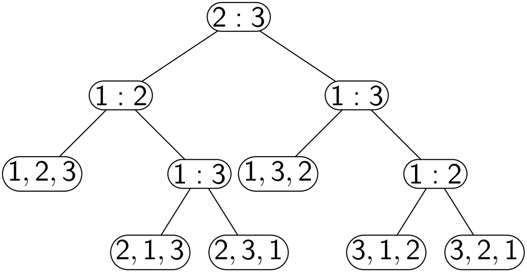
We label the root of the decision tree with $i : j$.

The left (right) subtree is obtained by continuing the algorithm under the assumption that $A[i] < A[j]$ $(A[i] > A[j])$.

Lower bound for the worst case

This yields a binary tree with $n!$ many leaves because every input permutation must lead to a different leaf.

Example: Here is a decision tree for sorting an array of length 3.



Lower bound for the worst case

Note: The depth (= max. number of edges on a path from the root to a leaf) of the decision tree is the maximal number of comparisons of the algorithm on an input array of length n.

A binary tree with N leaves has depth $\geq \log_2(N)$.

Stirling's formula (we only need $n! > \sqrt{2\pi n}(n/e)^n$) implies

$$\log_2(n!) \geq n\log_2(n) - \log_2(e)n + \Omega(\log n) \geq n\log_2(n) - 1,443n.$$

Thus, there exists an input array for which the algorithm makes at least $n\log_2(n) - 1,443n$ many comparisons. □

Lower bound for the average case

A comparison-based sorting algorithm even makes $n\log_2(n) - 2,443n$ many comparisons on almost all input permutations:

Theorem 4

For every comparison-based sorting algorithm the following holds: The portion of all permutations on which the algorithm makes at least

$$\log_2(n!) - n \geq n\log_2(n) - 2,443n$$

many comparisons is at least $1 - 2^{-n+1}$.

For the proof we need a simple lemma:

Lemma 5

Let $A \subseteq \{0,1\}^*$ *with* $|A| = N$, *and let* $1 \leq n < \log_2(N)$. *Then, at least* $(1 - 2^{-n+1})N$ *many words in* A *have length* $\geq \log_2(N) - n$.

Lower bound for the average case

Consider again the decision tree. It has $n!$ leaves, and every leaf corresponds to a permutation of the numbers $\{1, \ldots, n\}$.

Thus, each of the $n!$ many permutations can be represented by a word over the alphabet $\{0, 1\}$:

- 0 means: go in the decision tree to the left child.
- 1 means: go in the decision tree to the right child.

Lemma 5 $\rightsquigarrow$ the decision tree has at least $(1 - 2^{-n+1})n!$ many root-leaf paths of length $\geq \log_2(n!) - n \geq n \log_2(n) - 2,443n$. $\square$

Lower bound for the average case

Corollary

Every comparison-based sorting algorithm makes on average at least $n\log_2(n) - 2,443n$ many comparisons when sorting an array of length n (for n large enough).

Proof: Due to Theorem 4 at least

$$\begin{aligned}
(1-2^{-n+1})\cdot(\log_2(n!)-n)+2^{-n+1} &= \\
\log_2(n!) - n - \frac{\log_2(n!)-n-1}{2^{n-1}} &\geq \\
n\log_2(n) - 2,443n + \Omega(\log_2 n) - \frac{\log_2(n!)-n-1}{2^{n-1}} &\geq \\
n\log_2(n) - 2,443n &
\end{aligned}$$

many comparisons are done in the average. □

Quicksort

The Quicksort-algorithm (Tony Hoare, 1962):

- Choose an array-element $p = A[i]$ (the pivot element).
- Partitioning: Permute the array such that on the left (resp., right) of the pivot element p all elements are $\leq p$ (resp., $> p$) (needs $n-1$ comparisons).
- Apply the algorithm recursively to the subarrays to the left and right of the pivot element.

Critical: choice of the pivot elements.

- Running time is optimal, if the pivot element is the middle element of the array (median).
- Good choice in practice: median-out-of-three

Partitioning

First, we present a procedure for partitioning a subarray $A[\ell, \ldots, r]$ with respect to a pivot element $P = A[p]$, where $\ell < r$ and $\ell \leq p \leq r$.

The procedure returns an index $m \in \{\ell, \ldots, r\}$ with the following properties:

- $A[m] = P$
- $A[k] \leq P$ for all $\ell \leq k \leq m-1$
- $A[k] > P$ for all $m+1 \leq k \leq r$

Partitioning

Algorithm Partition

function partition($A[\ell \dots r]$: **array of integer**, p : **integer**) : **integer**
begin
    swap(p, r);
    $P := A[r]$;
    $i := \ell - 1$;
    for $j := \ell$ **to** $r - 1$ **do**
        if $A[j] \leq P$ **then**
            $i := i + 1$;
            swap(i, j)
        endif
    endfor
    swap($i + 1, r$)
    return $i + 1$
endfunction

Partitioning

The following invariants hold before every iteration of the **for**-loop:

- $A[r] = P$
- $A[k] \leq P$ for all $\ell \leq k \leq i$
- $A[k] > P$ for all $i + 1 \leq k \leq j - 1$

Thus, the following holds before the **return**-statement:

- $A[k] \leq P$ for all $\ell \leq k \leq i + 1$
- $A[k] > P$ for all $i + 2 \leq k \leq r$
- $A[i + 1] = P$

Note: partition($A[\ell \dots r]$) makes $r - \ell$ many comparisons.

Quicksort

Algorithm Quicksort

procedure quicksort($A[\ell \dots r]$: **array of integer**)
begin
  if $\ell < r$ **then**
    $p := r$;   (see next slide)
    $m :=$ partition($A[\ell \dots r]$, p);
    quicksort($A[\ell \dots m-1]$);
    quicksort($A[m+1 \dots r]$);
  endif
endprocedure

Quicksort

The running time of quicksort depends on the choices of the pivot elements.

The worst-case arises when after each call of partition$(A[\ell \dots r], p)$, one of the subarrays $(A[\ell \dots m-1]$ or $A[m+1 \dots r])$ is empty.

Better strategies to choose the pivot element $p \in \{\ell, \dots, r\}$:

- Let p be the index of the median of $A[\ell]$, $A[(\ell + r)$ div $2]$, $A[r]$
- Choose $p \in \{\ell, \dots, r\}$ uniformly at random ("randomized quicksort")

Worst-case running time: $\mathcal{O}(n^2)$.

Quicksort: average case analysis

Let $T(A)$ be the expected number of comparisons of randomized quicksort on input array A.

Lemma 6

If A and B are arrays of the same length then $T(A) = T(B)$.

Proof: Let A and B be arrays of length n. For a pivot index $1 \leq i \leq n$ let $L_i(A)$ and $R_i(A)$ be the resulting left and right subarrays of A. Then:

$$\begin{aligned} T(A) &= (n-1) + \frac{1}{n}\sum_{i=1}^{n}[T(L_i(A)) + T(R_i(A))] \\ &= (n-1) + \frac{1}{n}\sum_{i=1}^{n}[T(L_i(B)) + T(R_i(B))] = T(B) \end{aligned}$$

where $L_i(A) = L_i(B)$ and $R_i(A) = R_i(B)$ hold by induction hypothesis. □

Quicksort: average case analysis

Let $Q(n)$ be the expected number of comparisons on an input array of length n, satisfying the recurrence relation

$$Q(n) = (n-1) + \frac{1}{n}\sum_{i=1}^{n}[Q(i-1) + Q(n-i)].$$

Equivalently: $Q(n)$ is the expected number of comparisons of (deterministic) quicksort on a random input array where the rightmost element is always chosen as pivot.

Theorem 7

We have

$$Q(n) = 2(n+1)H(n) - 4n,$$

where $H(n) := \sum_{k=1}^{n} \frac{1}{k}$ *is the n-th harmonic number.*

Quicksort: average case analysis

Proof of Theorem 7:

For $n = 0$ we have $Q(0) = 0 = 2 \cdot 1 \cdot 0 - 4 \cdot 0$.

For $n = 1$ we have $Q(1) = 0 = 2 \cdot 2 \cdot 1 - 4 \cdot 1$.

For $n \geq 2$ we have:

$$\begin{aligned} Q(n) &= (n-1) + \frac{1}{n} \sum_{i=1}^{n} [Q(i-1) + Q(n-i)] \\ &= (n-1) + \frac{2}{n} \sum_{i=1}^{n} Q(i-1) \end{aligned}$$

We get:

$$nQ(n) = n(n-1) + 2 \sum_{i=1}^{n} Q(i-1)$$

Quicksort: average case analysis

Hence:

$$
\begin{aligned}
nQ(n) - (n-1)Q(n-1) &= n(n-1) + 2\sum_{i=1}^{n} Q(i-1) \\
&\quad -(n-1)(n-2) - 2\sum_{i=1}^{n-1} Q(i-1) \\
&= n(n-1) - (n-2)(n-1) + 2Q(n-1) \\
&= 2(n-1) + 2Q(n-1)
\end{aligned}
$$

We obtain:

$$
\begin{aligned}
nQ(n) &= 2(n-1) + 2Q(n-1) + (n-1)Q(n-1) \\
&= 2(n-1) + (n+1)Q(n-1)
\end{aligned}
$$

Quicksort: average case analysis

Dividing both sides by $n(n+1)$ gives:

$$\frac{Q(n)}{n+1} = \frac{2(n-1)}{n(n+1)} + \frac{Q(n-1)}{n}$$

Using induction on n we get:

$$\begin{aligned}
\frac{Q(n)}{n+1} &= \sum_{k=1}^{n} \frac{2(k-1)}{k(k+1)} \\
&= 2\sum_{k=1}^{n} \frac{(k-1)}{k(k+1)} \\
&= 2\Big(\sum_{k=1}^{n} \frac{k}{k(k+1)} - \sum_{k=1}^{n} \frac{1}{k(k+1)}\Big)
\end{aligned}$$

Quicksort: average case analysis

$$
\begin{aligned}
\frac{Q(n)}{n+1} &= 2\left[\sum_{k=1}^{n}\frac{1}{k+1} - \sum_{k=1}^{n}\frac{1}{k(k+1)}\right] \\
&= 2\left[\sum_{k=1}^{n}\frac{2}{k+1} - \sum_{k=1}^{n}\frac{1}{k}\right] \\
&= 2\left[2\left(\frac{1}{n+1} + H(n) - 1\right) - H(n)\right] \\
&= 2H(n) + \frac{4}{n+1} - 4.
\end{aligned}
$$

Quicksort: average case analysis

Finally, we get for $Q(n)$:

$$\begin{aligned} Q(n) &= 2(n+1)H(n) + 4 - 4(n+1) \\ &= 2(n+1)H(n) - 4n. \qquad \square \end{aligned}$$

One has $H(n) - \ln n \approx 0{,}57721\ldots =$ Euler's constant. Hence:

$$\begin{aligned} Q(n) &\approx 2(n+1)(0{,}58 + \ln n) - 4n \\ &\approx 2n \ln n - 2{,}8n \approx 1{,}38 n \log n - 2{,}8n. \end{aligned}$$

Theoretical optimum: $\log(n!) \approx n \log n - 1{,}44n$;
In the average, quicksort is only 38% worse than the optimum.

An average analysis of the median-out-of-three method yields $1{,}18n \log n - 2{,}2n$.
It is in the average only 18% worse than the optimum.

Heaps

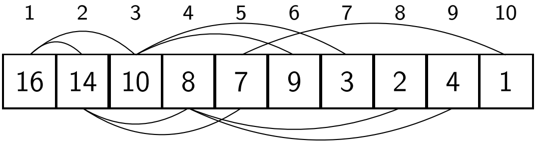
Definition 8

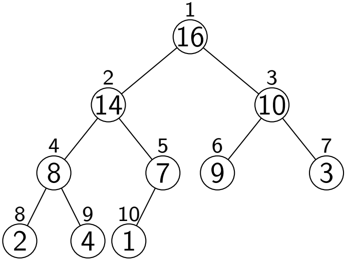
A (max-)heap is an array $a[1\dots n]$ with the following properties:

- $a[i] \geq a[2i]$ for all $i \geq 1$ with $2i \leq n$
- $a[i] \geq a[2i+1]$ for all $i \geq 1$ with $2i+1 \leq n$

Heaps

Example:



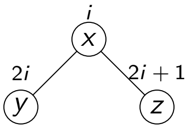


Sinking process

In a first step we will permute the entries of the array $a[1,\ldots,n]$ such that the heap condition is satisfied.

Assume that the subarray $a[i+1,\ldots,n]$ already satisfies the heap condition.

In order to enforce the heap condition also for i we let $a[i]$ sink:



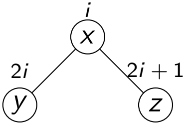
With 2 comparisons one can compute $\max\{x,y,z\}$.

Sinking process

In a first step we will permute the entries of the array $a[1, \ldots, n]$ such that the heap condition is satisfied.

Assume that the subarray $a[i + 1, \ldots, n]$ already satisfies the heap condition.

In order to enforce the heap condition also for i we let $a[i]$ sink:



With 2 comparisons one can compute $\max\{x, y, z\}$.

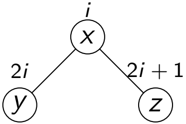
If x is the max., then the sinking process stops.

Sinking process

In a first step we will permute the entries of the array $a[1, \ldots, n]$ such that the heap condition is satisfied.

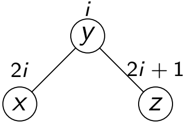
Assume that the subarray $a[i+1, \ldots, n]$ already satisfies the heap condition.

In order to enforce the heap condition also for i we let $a[i]$ sink:



With 2 comparisons one can compute $\max\{x, y, z\}$.

If y is the max., then x and y are swapped and we continue at $2i$.

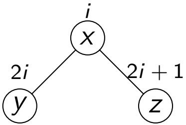


Sinking process

In a first step we will permute the entries of the array $a[1, \ldots, n]$ such that the heap condition is satisfied.

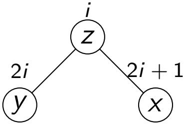
Assume that the subarray $a[i+1, \ldots, n]$ already satisfies the heap condition.

In order to enforce the heap condition also for i we let $a[i]$ sink:



With 2 comparisons one can compute $\max\{x, y, z\}$.

If z is the max., then x and z are swapped and we continue at $2i+1$.



Reheap

Algorithm Reheap

procedure reheap(i, n: **integer**) (∗ i is the root ∗)
var m: integer;
begin
  if $i \leq n/2$ **then**
    $m := \max\{a[i], a[2i], a[2i+1]\}$; (∗ 2 comparisons! ∗)
    if $(m \neq a[i]) \wedge (m = a[2i])$ **then**
      swap($i, 2i$); (∗ swap x, y ∗)
      reheap($2i, n$)
    elsif $(m \neq a[i]) \wedge (m = a[2i+1])$ **then**
      swap($i, 2i+1$); (∗ swap x, z ∗)
      reheap($2i+1, n$)
    endif
  endif
endprocedure

Building the heap

Algorithm Build Heap

procedure build-heap(n: **integer**)
begin
  for $i := \left\lfloor \frac{n}{2} \right\rfloor$ **downto** 1 **do**
    reheap(i, n)
  endfor
endprocedure

Invariant: Before the call of reheap(i, n) the subarray $a[i+1, \ldots, n]$ satisfies the heap condition.

Clearly, this holds for $i = \left\lfloor \frac{n}{2} \right\rfloor$.

Assume that the invariant holds for i.

Thus, the heap condition can only fail for i.

After the sinking process for $a[i]$, the heap condition also holds for i.

Time analysis for building the heap

Theorem 9

Built-heap runs in time $\mathcal{O}(n)$.

Proof: Sinking of $a[i]$ needs $2 \cdot$ (height of the subtree under $a[i]$) many comparisons.

We carry out the computation for $n = 2^k - 1$.

Then we have a complete binary tree of height $k - 1$.

There are

- 2^0 trees of height $k - 1$,
- 2^1 trees of height $k - 2$,

  $\vdots$
- 2^i trees of height $k - 1 - i$,

  $\vdots$
- 2^{k-1} trees of height 0.

Time analysis for building the heap

Hence, building the heap needs at most

$$\begin{aligned} 2 \cdot \sum_{i=0}^{k-1} 2^i (k-1-i) &= 2 \cdot \sum_{i=0}^{k-1} 2^{k-1-i} i \\ &= 2^k \cdot \sum_{i=0}^{k-1} i \cdot 2^{-i} \\ &\leq (n+1) \cdot \sum_{i \geq 0} i \cdot 2^{-i} \end{aligned}$$

many comparisons.

Claim: $\sum_{j \geq 0} j \cdot 2^{-j} = 2$

Proof of the claim: For every $|z| < 1$ we have

$$\sum_{j \geq 0} z^j = \frac{1}{1-z}.$$

Time analysis for building the heap

Taking derivations yields

$$\sum_{j \geq 0} j \cdot z^{j-1} = \frac{1}{(1-z)^2},$$

and hence

$$\sum_{j \geq 0} j \cdot z^{j} = \frac{z}{(1-z)^2}.$$

Setting $z = 1/2$ yields

$$\sum_{j \geq 0} j \cdot 2^{-j} = 2.$$

$\square$

Standard Heapsort (W. J. Williams, 1964)

Algorithm Heapsort

procedure heapsort(n: **integer**)
begin
  build-heap(n)
  for $i := n$ **downto** 2 **do**
    swap($1, i$);
    reheap($1, i-1$)
  endfor
endprocedure

Theorem 10

Standard Heapsort sorts an array with n elements and needs $2n \log_2 n + \mathcal{O}(n)$ comparisons.

Standard Heapsort

Proof:

Correctness: After build-heap(n), $a[1]$ is the maximal element of the array.

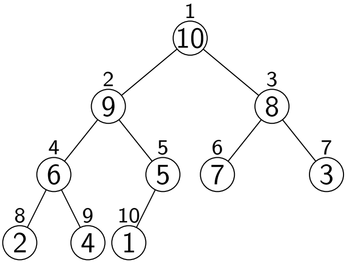
This element will be moved with swap$(1, n)$ to its correct position (n).

By induction, the subarray $a[1, \ldots, n-1]$ will be sorted in the remaining steps.

Running time: Building the heap needs $\mathcal{O}(n)$ comparison. Each of the remaining $n-1$ many reheap-calls needs at most $2\log_2 n$ comparisons. □

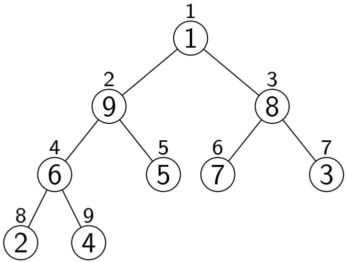
Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10



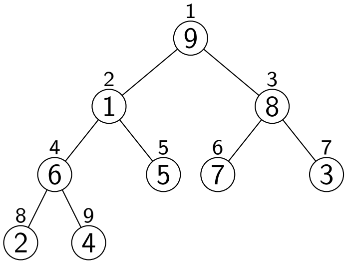
Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
									10

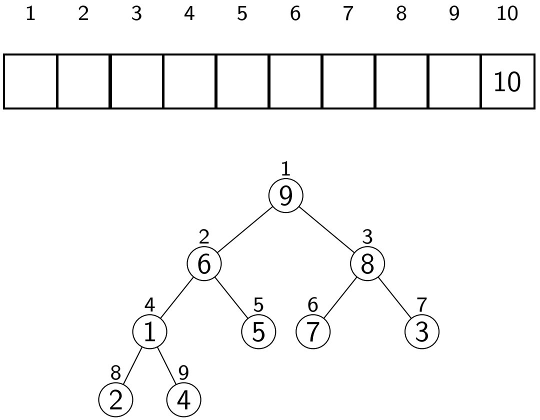


Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
									10

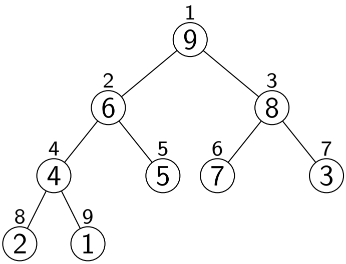


Example for Standard Heapsort



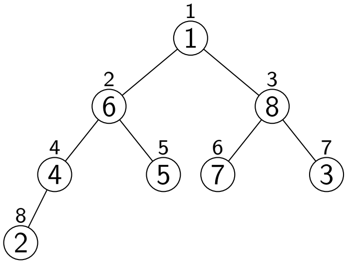
Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
									10



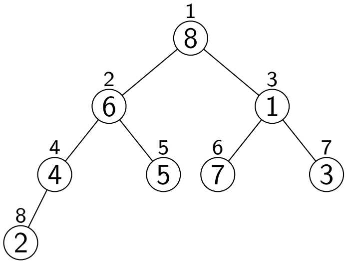
Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
								9	10



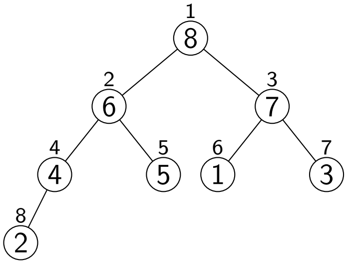
Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
								9	10



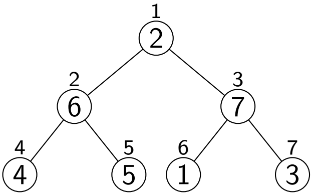
Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
								9	10



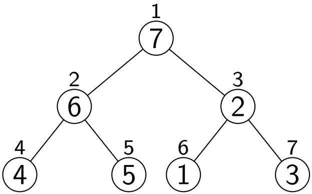
Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
							8	9	10



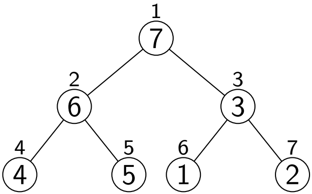
Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
							8	9	10



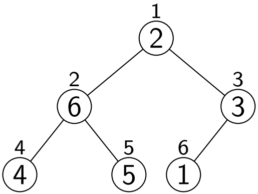
Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
							8	9	10



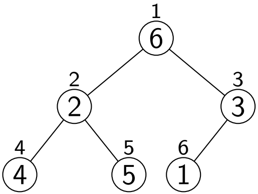
Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
						7	8	9	10



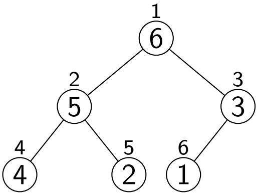
Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
						7	8	9	10



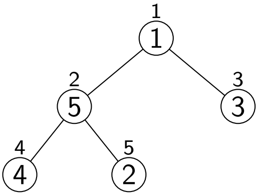
Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
						7	8	9	10



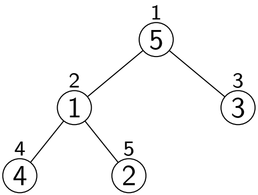
Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
					6	7	8	9	10



Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
					6	7	8	9	10



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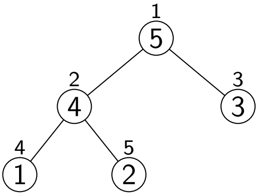
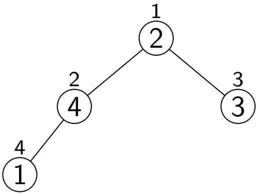
1	2	3	4	5	6	7	8	9	10
					6	7	8	9	10



Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
				5	6	7	8	9	10



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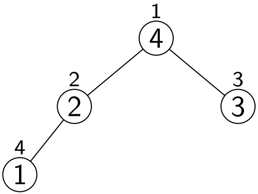
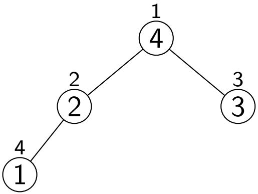
1	2	3	4	5	6	7	8	9	10
				5	6	7	8	9	10



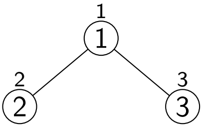
Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
				5	6	7	8	9	10



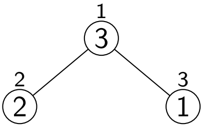
Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
			4	5	6	7	8	9	10



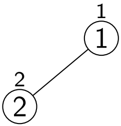
Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
			4	5	6	7	8	9	10



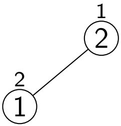
Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
		3	4	5	6	7	8	9	10



Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
		3	4	5	6	7	8	9	10



Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
	2	3	4	5	6	7	8	9	10

1

1

Example for Standard Heapsort

1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10

Bottom-Up Heapsort

Remark: An analysis of the average case complexity of Heapsort yields $2n \log_2 n$ many comparisons in the average. Hence, standard Heapsort cannot compete with Quicksort.

Bottom-up Heapsort needs significantly fewer comparisons.

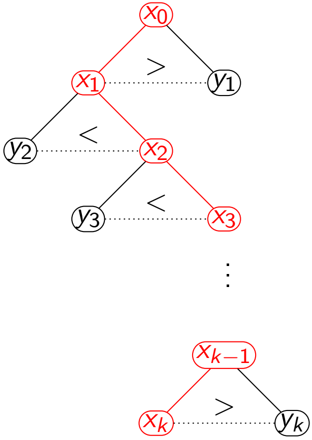
After swap$(1, i)$ one first determines the potential path from the root to a leaf along which the elemente $a[i]$ will sink; the sink path.

For this, one follows the path that always goes to the larger child. This needs at most $\log n$ instead of $2 \log_2 n$ comparisons.

In most cases, $a[i]$ will sink deep into the heap. It is therefore more efficient to compute the actual position of $a[i]$ on the sink path bottom-up.

The hope is that the bottom-up computations need in total only $\mathcal{O}(n)$ comparisons.

The sink path



Elements will sink along the path $[x_0, x_1, x_2, \dots, x_{k-1}, x_k]$ which can be computed with only $\log_2 n$ comparisons.

Finding the right position on the sink path

We now compute the right position p on the sink path starting from the leaf and going up.

If this position p is found, then all elements $x_0, \ldots, x_p$ have to be rotated cyclically (x_0 goes to the position of x_p, and every $x_1, \ldots, x_p$ moves up one position).

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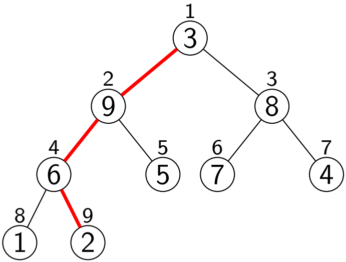

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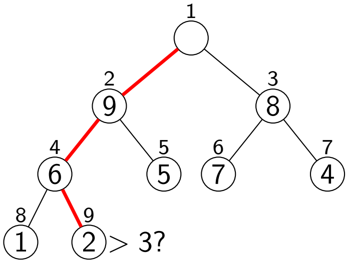

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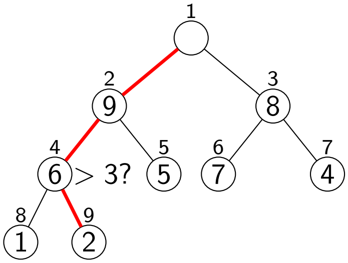

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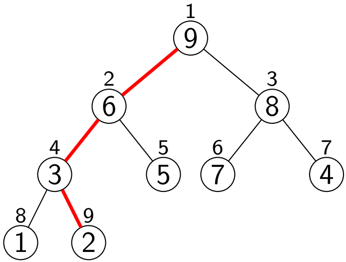


Jensen's Inequality

Let $f : D \to \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}$ is an interval.

- f is convex if for all $x, y \in D$ and all $0 \leq \lambda \leq 1$, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.
- f is concave if for all $x, y \in D$ and all $0 \leq \lambda \leq 1$, $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$.

Jensen's inequality

If f is convex, then for all $x_1, \ldots, x_n \in D$ and all $\lambda_1, \ldots, \lambda_n \geq 0$ with $\lambda_1 + \cdots + \lambda_n = 1$:

$$f\left(\sum_{i=1}^{n} \lambda_i \cdot x_i\right) \leq \sum_{i=1}^{n} \lambda_i \cdot f(x_i).$$

If f is concave, then for all $x_1, \ldots, x_n \in D$ and all $\lambda_1, \ldots, \lambda_n \geq 0$ with $\lambda_1 + \cdots + \lambda_n = 1$:

$$f\left(\sum_{i=1}^{n} \lambda_i \cdot x_i\right) \geq \sum_{i=1}^{n} \lambda_i \cdot f(x_i).$$

Average Analysis of Heapsort

Theorem 11

Standard heapsort makes on a portion of at least $1 - 2^{-(n-1)}$ *many input permutations at least* $2n\log_2(n) - \mathcal{O}(n)$ *many comparisons.*
Bottom-up heapsort makes on a portion of at least $1 - 2^{-(n-1)}$ *many input permutations at most* $n\log_2(n) + \mathcal{O}(n)$ *many comparisons.*

Proof: information-theoretic argument

A sorting algorithm computes from a permutation of $[1, \ldots, n]$ the sorted list $[1, \ldots, n]$.

One can specify (or encode) the input permutation by running the algorithm and in addition output information in form of a $\{0, 1\}$-string that allows us to run the algorithm backwards starting with the output permutation $[1, \ldots, n]$.

Average Analysis of Heapsort

In the case of standard heapsort: we output the sink paths, i.e., every time an element is swapped with the left (resp., right) child, we output a 0 (resp., 1). This makes heapsort reversible.

But: We have to know when one sink path (a $\{0,1\}$-string) stops and the next sink path starts.

Alternative 1: We encode a string $w = a_1 a_2 \cdots a_{t-1} a_t \in \{0,1\}^*$ by

$$c_1(w) = a_1 0 a_2 0 \cdots a_{t-1} 0 a_t 1.$$

Note: $|c_1(w)| = 2|w|$.

Alternative 2: We encode a string $w = a_1 a_2 \cdots a_{t-1} a_t \in \{0,1\}^*$ by

$$c_2(w) = c_1(\text{binary representation of } t) a_1 \cdots a_t$$

Thus, $|c_2(w)| = |w| + 2\log_2(|w|)$.

Average Analysis of Heapsort

Note: $c_2(\varepsilon) = 01$, since $0 =$ binary representation of the number 0.

We encode the sink path $w = a_1 a_2 \cdots a_t \in \{0,1\}^*$ by

$$c_2'(w) = c_1(\text{binary representation of } \log_2(n) - t) a_1 \cdots a_t.$$

Note: $t \leq \log_2(n)$, because every sink path has length $\leq \log_2 n$.

Our proof showing that building the heap only needs $\mathcal{O}(n)$ many comparisons also shows: In phase 1, we will output a $\{0,1\}$-string of length $\mathcal{O}(n)$.

We now analyse the $\{0,1\}$-string produced in phase 2.

Average Analysis of Heapsort

Let $t_1, \ldots, t_n$ be the lengths of the sink paths during phase 2.

Hence, we produce in phase 2 a $\{0,1\}$-string of length

$$\sum_{i=1}^{n}(t_i + 2\log_2(\log_2(n) - t_i)) = \sum_{i=1}^{n} t_i \; + \; 2\sum_{i=1}^{n} \log_2(\log_2(n) - t_i)).$$

Define the average

$$\bar{t} = \frac{\sum_{i=1}^{n} t_i}{n}.$$

The function f with $f(x) = \log_2(\log_2(n) - x)$ is concave on $(-\infty, \log_2(n))$.

Jensen's inequality (slide 62) implies:

$$\log_2(\log_2(n) - \bar{t}) \geq \sum_{i=1}^{n} \frac{1}{n} \cdot \log_2(\log_2(n) - t_i)).$$

Average Analysis of Heapsort

Therefore:

$$\sum_{i=1}^{n} t_i \; + \; 2\sum_{i=1}^{n} \log_2(\log_2(n) - t_i)) \quad \leq \quad n\bar{t} + 2n\log_2(\log_2(n) - \bar{t}).$$

To sum up: The input permutation σ on $[1, \ldots, n]$ can be encoded by a $\{0,1\}$-string of length

$$I(\sigma) \leq cn + n\bar{t} + 2n\log_2(\log_2(n) - \bar{t}),$$

where c is a constant (for phase 1).

Lemma 5 implies

$$cn + n\bar{t} + 2n\log_2(\log_2(n) - \bar{t}) \geq I(\sigma) \geq \log_2(n!) - n \geq n\log_2(n) - 2{,}443n$$

for at least $(1 - 2^{-n+1})n!$ many input permutations.

With $d = 2{,}443 + c$ we get:

$$\bar{t} \geq \log_2(n) - 2\log_2(\log_2(n) - \bar{t}) - d. \tag{1}$$

Average Analysis of Heapsort

Since $\bar{t} \geq 0$ we obtain

$$\bar{t} \geq \log_2(n) - 2\log_2(\log_2(n)) - d. \tag{2}$$

From (1) and (2) we get the better estimate

$$\bar{t} \geq \log_2(n) - 2\log_2(2\log_2(\log_2(n)) + d) - d. \tag{3}$$

This estimate can be again applied to (1), and so on.

In general, we get for all $i \geq 1$:

$$\bar{t} \geq \log_2(n) - \alpha_i - d,$$

where $\alpha_1 = 2\log_2(\log_2(n))$ and $\alpha_{i+1} = 2\log_2(\alpha_i + d)$.
(proof by induction on $i \geq 1$)

Average Analysis of Heapsort

For all $x \geq \max\{10, d\}$ we have:

$$2\log_2(x+d) \leq 2\log_2(2x) = 2\log_2(x) + 2 \leq 0,9 \cdot x.$$

Hence, as long as $\alpha_i \geq \max\{10, d\}$ holds, we have $\alpha_{i+1} \leq 0,9 \cdot \alpha_i$.

Therefore, there exists a constant α with

$$\bar{t} \geq \log_2(n) - \alpha - d. \tag{4}$$

Thus, for at least $(1 - 2^{-n+1})n!$ many input permutations we have

$$\sum_{i=1}^{n} t_i \geq n \log_2 n - \mathcal{O}(n).$$

Average Analysis of Heapsort

The statement of Theorem 11 for standard Heapsort follows easily:

In phase 2, standard Heapsort makes $2\sum_{i=1}^{n} t_i$ many comparisons.

Hence, standard-Heapsort makes for at least $(1-2^{-n+1})n!$ many input permutations at least $2n\log_2 n - \mathcal{O}(n)$ many comparisons.

Bottom-up heapsort makes in phase 2 at most

$$n\log_2(n) + \sum_{i=1}^{n}(\log_2(n) - t_i) = 2n\log_2(n) - \sum_{i=1}^{n} t_i$$

many comparisons.

Hence, bottom-up Heapsort makes for at least $(1-2^{-n+1})n!$ many input permutations at most

$$\mathcal{O}(n) + 2n\log_2(n) - \sum_{i=1}^{n} t_i \leq n\log_2(n) + \mathcal{O}(n)$$

many comparisons. $\square$

Variant by Svante Carlsson, 1986

One can show that bottom-up Heapsort makes in the worst case at most $1.5n \log n + \mathcal{O}(n)$ many comparisons.

Carlsson proposed to determine the correct position on the sink path using binary search.

This yields a worst-case bound of $n \log n + \mathcal{O}(n \log \log n)$ many comparison.

On the other hand, in practice binary search on the sink path does not seem to pay off.

Counting Sort

Recall: The lower bound of $\Omega(n \log n)$ only holds for comparison-based sorting algorithms.

If we make further assumptions on the array elements, we can sort in time $\mathcal{O}(n)$.

Assumption: The array elements $A[1], \ldots, A[n]$ are natural numbers in the range $[0, k]$.

Counting sort (see next slide) sorts under this assumption in time $\mathcal{O}(k + n)$.

Hence, if $k \in \mathcal{O}(n)$, then counting sort works in linear time.

Counting Sort

Algorithm Counting Sort

procedure counting-sort(array $A[1,n]$ with $A[1], \ldots A[n] \in [0,k]$)
begin
  var Arrays $C[0,k]$, $B[1,n]$
  for $i := 0$ **to** k **do**
    $C[i] := 0$
  for $i := 1$ **to** n **do**
    $C[A[i]] := C[A[i]] + 1$
  for $i := 1$ **to** k **do**
    $C[i] := C[i] + C[i-1]$
  for $i := n$ **downto** 1 **do**
    $B[C[A[i]]] := A[i]$;
    $C[A[i]] := C[A[i]] - 1$
endprocedure

Counting Sort

After the first three for-loops, $C[i]$ is the number of array entries that are $\leq i$.

The statement $B[C[A[i]]] := A[i]$ puts the array element $A[i]$ at the right position $C[A[i]]$.

Remark: Counting sort is a stable sorting algorithm.

This means: If $A[i] = A[j]$ for $i < j$, then in the sorted array B the array entry $A[i]$ is to the left of $A[j]$.

This is relevant if the array entries consist of (i) keys that are used for sorting and (ii) additional informations.

Stability of counting sort will be needed for radix sort on the next slide.

Radix Sort

We use counting sort to sort an array $A[1, n]$, where $A[1], \ldots, A[n]$ are d-ary numbers in base k (where the least significant digit is the left most digit).

Radix sort sorts such an array in time $\mathcal{O}(d(n+k))$.

If in addition $d \in \mathcal{O}(1)$ and $k \in \mathcal{O}(n)$ (which means that we can represent number of size $\mathcal{O}(n^d)$), then radix sort works in linear time.

Algorithm Radix Sort

procedure radix sort(array $A[1, n]$ with $A[1], \ldots A[n]$)
begin
  for $i := 1$ **to** d **do**
    sort the array A with counting sort with respect to the i-th digit.
  endfor
endprocedure

Computation of the Median

Input: array $a[1, \dots, n]$ of numbers and $1 \leq k \leq n$.

Output: k-th smallest element, i.e., the number $m \in \{a[i] \mid 1 \leq i \leq n\}$ such that

$$|\{i \mid a[i] < m\}| \leq k - 1 \quad \text{and} \quad |\{i \mid a[i] > m\}| \leq n - k$$

Special case: $k = \lceil n/2 \rceil \rightsquigarrow$ median

Naive approach:

- sort the array a in time $\mathcal{O}(n \log n)$,
- output the k-th element of the sorted array.

Median of the medians

Goal: Compute the k-th smallest element in linear time.

Idea: Compute a pivot element (as in quick sort) as the median of the medians of blocks of length 5.

- We split the array in blocks of length 5.
- For each block we compute the median (6 comparisons are sufficient).
- Compute recursively the median p of the array of medians and take p as the pivot element.

  Number of comparisons: $T(\frac{n}{5})$.

Quick sort step

Partition the array with the pivot element p such that for suitable positions $m_1 < m_2$ we have:

$$\begin{array}{lcll} a[i] & < & p & \text{for } 1 \leq i \leq m_1 \\ a[i] & = & p & \text{for } m_1 < i \leq m_2 \\ a[i] & > & p & \text{for } m_2 < i \leq n \end{array}$$

Number of comparisons: $\leq n$.

Case distinction:

1. $k \leq m_1$: Search for the k-th element recursively in $a[1], \ldots, a[m_1]$.
2. $m_1 < k \leq m_2$: Return p.
3. $k > m_2$: Search for the $(k - m_2)$-th element in $a[m_2 + 1], \ldots, a[n]$.

30 – 70 splitting

The choice of the pivot element as the median of the medians (of blocks of length 5) ensures the following inequalites for m_1, m_2:

$$\frac{3}{10}n \leq m_2 \qquad \text{and} \qquad m_1 \leq \frac{7}{10}n$$

Therefore, the recursive step needs at most $T(\frac{7n}{10})$ comparisons.

Total time

$T(n)$ is the total number of comparisons for an array of length n.

We get the following recurrence for $T(n)$:

$$T(n) \leq T\left(\lceil \frac{n}{5} \rceil\right) + T\left(\lceil \frac{7n}{10} \rceil\right) + \mathcal{O}(n)$$

The master theorem II gives $T(n) \in \mathcal{O}(n)$.

Master theorem II

Theorem 12 (Master theorem II)

Let $\alpha_0, \ldots, \alpha_r > 0$, $\sum_{i=0}^{r} \alpha_i < 1$ *and assume that for a constant* c,

$$t(n) \leq \left(\sum_{i=0}^{r} t(\lceil \alpha_i n \rceil) \right) + c \cdot n.$$

Then we have $t(n) \in \mathcal{O}(n)$.

Proof of the master theorem II

Choose $\varepsilon > 0$ and $n_0 > 0$ such that

$$\sum_{i=0}^{r} \lceil \alpha_i n \rceil \le (\sum_{i=0}^{r} \alpha_i) \cdot n + (r+1) \le (1-\varepsilon) n$$

for all $n \ge n_0$.

Choose γ such that $c \le \gamma\varepsilon$ and $t(n) \le \gamma n$ for all $n < n_0$.

By induction we get for all $n \ge n_0$:

$$\begin{aligned} t(n) &\le \left(\sum_{i=0}^{r} t(\lceil \alpha_i n \rceil) \right) + cn \\ &\le \left(\sum_{i=0}^{r} \gamma \lceil \alpha_i n \rceil \right) + cn \qquad \text{(induction)} \\ &\le (\gamma(1-\varepsilon) + c) n \\ &\le \gamma n \end{aligned}$$

$\square$

More precise analysis

$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + \frac{6n}{5} + \frac{2n}{5},$$

where:

- $\frac{6n}{5}$ is the number of comparisons to compute the medians of the blocks of length 5.
- $\frac{2n}{5}$ is the number of comparisons for the partitioning step.

This yields the bound $T(n) \leq 16n$:

With $\frac{1}{5} + \frac{7}{10} = \frac{9}{10}$ we get $T(n) \leq T(\frac{9n}{10}) + \frac{8n}{5}$ and hence $T(n) \leq 10 \cdot \frac{8n}{5} = 16n$.

Quick select

Quick select is a randomized algorithm for computing the median:

Algorithm

function quickselect($A[\ell \ldots r]$: **array of integer**, k : **integer**) : **integer**
begin
  if $\ell = r$ **then return** $A[\ell]$
  else
    $p := \mathrm{random}(\ell, r)$;
    $m :=$ partition($A[\ell \ldots r]$, p);
    $k' := (m - \ell + 1)$;
    if $k = k'$ **then return** $A[m]$
    elsif $k < k'$ **then return** quickselect($A[\ell \ldots m-1]$, k)
    else return quickselect($A[m+1 \ldots r]$, $k - k'$)
    endif
  endif
endfunction

Analysis of quick select

Let $Q(n)$ be the average number of comparisons that quick select is doing for an array with n elements.

We have:

$$Q(n) \leq (n-1) + \frac{1}{n}\sum_{i=1}^{n} Q(\max\{i-1, n-i\}),$$

where:

- $(n-1)$ is the number of comparisons for partitioning the array, and
- $Q(\max\{i-1, n-i\})$ is the (maximal) average number of comparisons for a recursive call on *one* of the two subarrays.

Here, we make the pessimistic assumption that we continue searching in the larger subarray.

Analysis of quick select

We have

$$\begin{aligned} Q(n) &\leq (n-1) + \frac{1}{n}\sum_{i=1}^{n} Q(\max\{i-1, n-i\}) \\ &= (n-1) + \frac{1}{n}\left(\sum_{i=\lceil \frac{n}{2}\rceil}^{n-1} Q(i) + \sum_{i=\lfloor \frac{n}{2}\rfloor}^{n-1} Q(i)\right) \end{aligned}$$

Claim: $Q(n) \leq 4 \cdot n$:

Proof by induction on n: OK for $n = 1$.

Let $n \geq 2$ and let $Q(i) \leq 4 \cdot i$ for all $i < n$.

Analysis of quick select

Case 1: n is even.

$$
\begin{aligned}
Q(n) &\le (n-1) + \frac{2}{n}\sum_{i=\frac{n}{2}}^{n-1} Q(i) \\
&\le (n-1) + \frac{8}{n}\sum_{i=\frac{n}{2}}^{n-1} i \\
&= (n-1) + \frac{8}{n}\left(\frac{(n-1)n}{2} - \frac{(\frac{n}{2}-1)\frac{n}{2}}{2}\right) \\
&= (n-1) + 4(n-1) - n + 2 \\
&= 4n - 3 \le 4n
\end{aligned}
$$

Analysis of quick select

Case 2: n is odd.

$$
\begin{aligned}
Q(n) &\leq (n-1) + \frac{2}{n} \sum_{i=\lceil \frac{n}{2} \rceil}^{n-1} Q(i) \; + \frac{1}{n} Q\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \\
&\leq (n-1) + \frac{8}{n} \sum_{i=\lceil \frac{n}{2} \rceil}^{n-1} i \; + 2 \\
&= (n-1) + \frac{8}{n} \cdot \left(\frac{(n-1)n}{2} - \frac{\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) \left\lceil \frac{n}{2} \right\rceil}{2} \right) + 2 \\
&\leq (n-1) + 4(n-1) - n - 2 + 2 \\
&= 4n - 5 \leq 4 \cdot n.
\end{aligned}
$$

Overview

- Matroids
- Kruskal's algorithm for spanning trees
- Dijkstra's algorithm for shortest paths

Greedy algorithms

Algorithms that take in each step the locally best optimal choice are called greedy.

For some problems this yields a globally optimal solution.

Problems where greedy algorithms always find an optimal solution can be characterized via the notion of a matroid.

Spanning subtrees

Let $G = (V, E)$ be a finite undirected graph.

A path from $u \in V$ to $v \in V$ is a sequence of nodes $(u_1, u_2, \dots, u_n)$ with $u_1 = u$, $u_n = v$ and $(u_i, u_{i+1}) \in E$ for all $1 \leq i \leq n-1$.

G is connected if for all $u, v \in V$ there exists a path from u to v.

A circuit is a path $(u_1, u_2, \dots, u_n)$ with $n \geq 3$, $u_i \neq u_j$ for all $1 \leq i < j \leq n$ and $(u_n, u_1) \in E$.

G is a tree if it is connected and has no circuits.

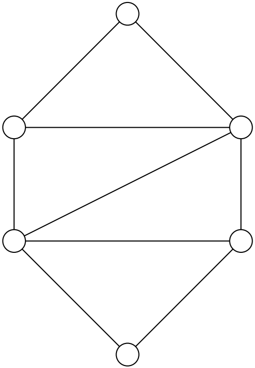
Exercise: for every tree $T = (V, E)$ we have $|E| = |V| - 1$.

Let $G = (V, E)$ be connected. A spanning subtree of G is a subset $F \subseteq E$ of edges such that (V, F) is a tree.

Exercise: every connected graph has a spanning subtree.

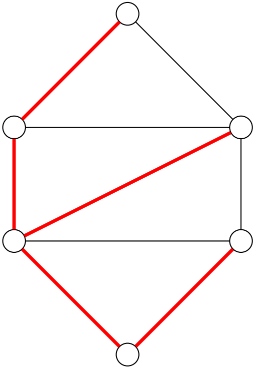
Spanning subtrees

Example:



Spanning subtrees

Example:



Maximum spanning subtrees

Let $G = (V, E)$ be again connected, and let $w\colon E \to \mathbb{R}$ be a weight function.

The weight of a spanning subtree $F \subseteq E$ is

$$w(F) = \sum_{e \in F} w(e).$$

Goal: Compute a spanning subtree of maximal weight.

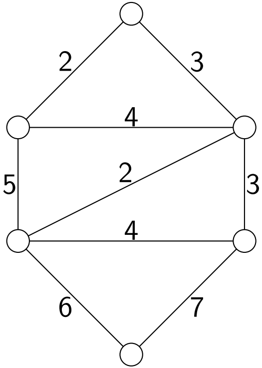
Kruskal's algorithm

Algorithm Kruskals algorithm

procedure kruskal (edge-weighted connected graph (V, E, w))
begin
  sort E by decreasing weights $e_1, e_2, \dots, e_n$ with
  $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_n)$
  $F := \emptyset$
  for $k := 1$ **to** n **do**
    if e_k connects two different connected components of (V, F) **then**
      $F := F \cup \{e_k\}$
  endfor
  return F
endprocedure

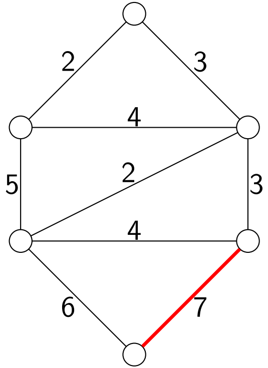
Kruskal's algorithm

Example for Kruskal's algorithm:



Kruskal's algorithm

Example for Kruskal's algorithm:



Kruskal's algorithm

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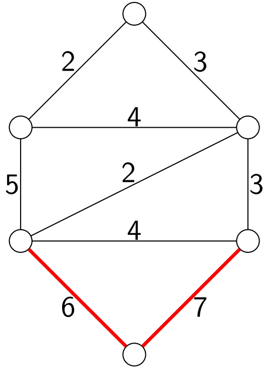

Kruskal's algorithm

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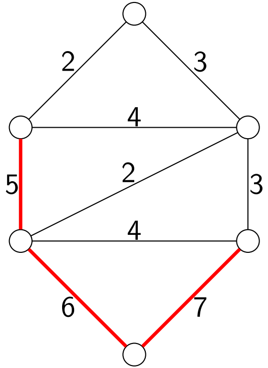

Kruskal's algorithm

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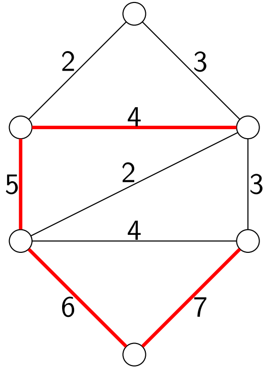

Kruskal's algorithm

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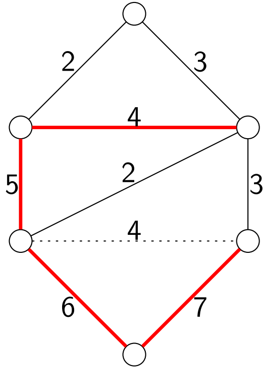

Kruskal's algorithm

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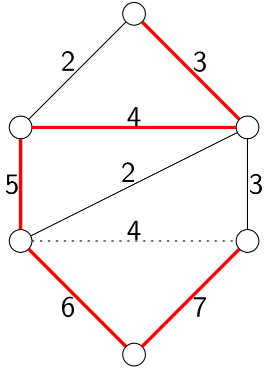

Kruskal's algorithm

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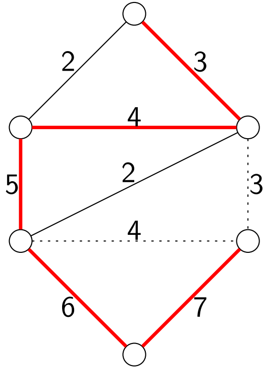

Kruskal's algorithm

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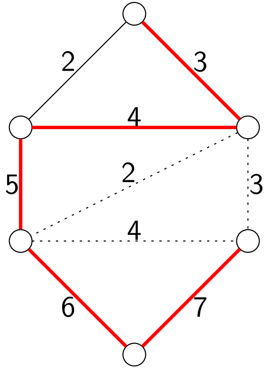
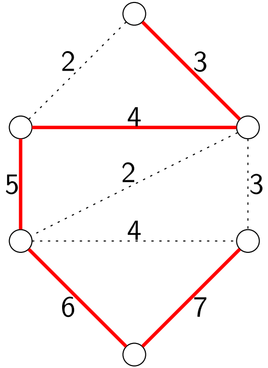


Kruskal's algorithm

Example for Kruskal's algorithm:



Running time of Kruskal's algorithm

Note: Since G is connected, we have $|V| - 1 \leq |E| \leq |V|^2$.

Sorting the edges by weight needs time $\mathcal{O}(|E| \log |E|) = \mathcal{O}(|E| \log |V|)$.

The connected components of (V, F) can be maintained by a partition of the node set V.

We start with the partition $\{\{v\} \mid v \in V\}$.

In every iteration of the **for**-loop ($|E|$ many) we test whether the end points of the edge e_k belong to different sets A, B of the partition.

If this holds, then we replace in the partition the sets A and B by the set $A \cup B$.

For this, we will later develop a so-called union-find data structure, which realizes the above operations in total time $\mathcal{O}(|E| \cdot \alpha(|V|))$ for an extremely slow-growing function α.

This gives the running time $\mathcal{O}(|E| \log |V|)$ for Kruskal's algorithm.

Optimization problems

Let E be a finite set and $U \subseteq 2^E$ a set of subsets of E.

A pair (E, U) is a subset system, if the following holds:

- $\emptyset \in U$
- If $A \subseteq B \in U$ then $A \in U$ as well.

A set $A \in U$ is maximal (with respect to $\subseteq$) if for all $B \in U$ the following holds: if $A \subseteq B$, then $A = B$.

The optimization problems associated with (E, U) is:

- Input: A weight function $w \colon E \to \mathbb{R}$
- Output: A maximal set $A \in U$ with $w(A) \geq w(B)$ for all maximal sets $B \in U$, where

$$w(C) = \sum_{a \in C} w(a)$$

We call A an optimal solution.

Optimization problems

In order to solve such optimization problems, one can try to use the following generic greedy algorithm:

Algorithm generic greedy algorithm

procedure find-optimal (subset system (E, U), $w : E \to \mathbb{R}$)
begin
  order set E by descending weights as $e_1, e_2, \ldots, e_n$ with
  $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_n)$
  $T := \emptyset$
  for $k := 1$ **to** n **do**
    if $T \cup \{e_k\} \in U$ **then** $T := T \cup \{e_k\}$
  endfor
  return T
endprocedure

Matroids

Note: The solution computed by the generic greedy algorithm is always a maximal subset.

Unfortunately there exist subset systems for which the generic greedy algorithm does not find an optimal solution (will be shown later).

A subset system (E, U) is a matroid, if the following property (exchange property) holds:

$$\forall A, B \in U : |A| < |B| \implies \exists x \in B \setminus A : A \cup \{x\} \in U$$

Remark: If (E, U) is a matroid, then all maximal sets in U have the same cardinality.

Example: Let E be a finite set and $k \leq |E|$. Then

$$(E, \{A \subseteq E \mid |A| \leq k\})$$

is a matroid.

Matroid of circuit-free edge sets

Lemma 13

The subset system $(E, \{A \subseteq E \mid (V, A) \text{ has no circuit}\})$ *is a matroid.*

Proof: Let $A, B \subseteq E$ be edge sets without circuits such that $|A| < |B|$.

Let $V_1, V_2 \ldots, V_n$ be the connected components of (V, A).

We have $|A| = \sum_{i=1}^{n}(|V_i| - 1)$, because the subtree of (V, A) induced by V_i is a tree and therefore has $|V_i| - 1$ many edges.

For every edge $e = (u, v) \in B$ one of the following two cases holds:

1. There is $1 \le i \le n$ with $u, v \in V_i$.
2. There are $i \neq j$ with $u \in V_i$ and $v \in V_j$.

But in B there can exist at most $\sum_{i=1}^{n}(|V_i| - 1) = |A|$ many edges of type 1 (otherwise B would contain a circuit in one of the sets V_i).

Hence, there exists an edge $e \in B$, which connects two connected components of (V, A). Thus, $A \cup \{e\}$ contains no circuit. $\square$

Matroids

Theorem 14

Let (E, U) be a subset system. The generic greedy algorithm computes for every weight function $w \colon E \to \mathbb{R}$ an optimal solution if and only if (E, U) is a matroid.

Proof: First assume that (E, U) is a matroid.

Let $w \colon E \to \mathbb{R}$ be a weight function and let $E = \{e_1, e_2, \ldots, e_n\}$ with

$$w(e_1) \geq w(e_2) \geq \cdots \geq w(e_n).$$

Let $T = \{e_{i_1}, \ldots, e_{i_k}\}$ with $i_1 < i_2 < \cdots < i_k$ the solution computed by the generic greedy algorithm.

Assumption: There exists a maximal set $S = \{e_{j_1}, \ldots, e_{j_l}\} \in U$ with $w(S) > w(T)$, where $j_1 < j_2 < \cdots < j_l$.

Since (E, U) is a matroid, we have $k = l$.

Matroids

Since $w(S) > w(T)$, there exists $1 \leq p \leq k$ with $w(e_{j_p}) > w(e_{i_p})$.

Since the weights were sorted in descending order, we must have $j_p < i_p$.

We now apply the exchange property to the sets

$$A = \{e_{i_1}, \ldots, e_{i_{p-1}}\} \in U \quad \text{and} \quad B = \{e_{j_1}, \ldots, e_{j_p}\} \in U.$$

Since $|A| < |B|$, there exists an element $e_{j_q} \in B \setminus A$ with $A \cup \{e_{j_q}\} \in U$.

We get $j_q \leq j_p < i_p$.

But then, the generic greedy algorithm would have put e_{j_q} into the solution, which is a contradiction.

Matroids

Now assume that (E, U) is not a matroid, i.e., the exchange property does not hold.

Let $A, B \in U$ with $|A| < |B|$ such that for all $b \in B \setminus A$: $A \cup \{b\} \notin U$.

Let $r = |B|$ and hence $|A| \leq r - 1$.

Define the weight function $w \colon E \to \mathbb{R}$ as follows:

$$w(x) = \begin{cases} r+1 & \text{for } x \in A \\ r & \text{for } x \in B \setminus A \\ 0 & \text{otherwise} \end{cases}$$

The generic greedy algorithm must compute a solution T with $A \subseteq T$ and $T \cap (B \setminus A) = \emptyset$.

We get $w(T) = (r+1) \cdot |A| \leq (r+1)(r-1) = r^2 - 1$.

Let $S \in U$ be a maximal subset with $B \subseteq S$.

We get $w(S) \geq w(B) \geq r^2$. $\square$

Shortest paths

Another example for a greedy strategy: Computation of shortest paths in an edge-weighted directed graph $G = (V, E, \gamma)$.

- V is the set of nodes
- $E \subseteq V \times V$ is the set of edges, where $(x, x) \notin E$ for all $x \in V$.
- $\gamma : E \to \mathbb{N}$ is the weight function.

Weight of a path $(v_0, v_1, v_2, \ldots, v_n)$: $\displaystyle\sum_{i=0}^{n-1} \gamma(v_i, v_{i+1})$

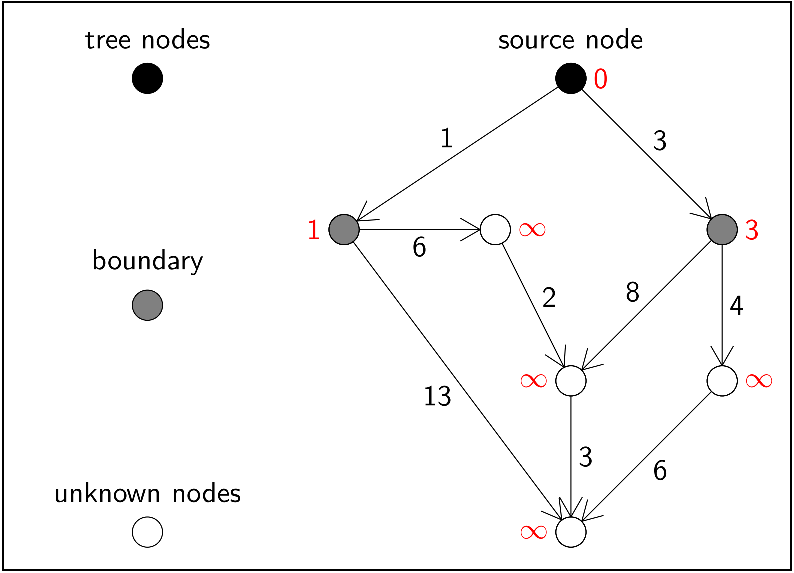
For $u, v \in V$, $d(u, v)$ denotes the minimum of the weight of all paths from u to v ($d(u, v) = \infty$ if such a path does not exist, and $d(u, u) = 0$).

Goal: Given $G = (V, E, \gamma)$ and a source node $u \in V$, compute for every $v \in V$ a path $u = v_0, v_1, v_2, \ldots, v_{n-1}, v_n = v$ with minimal weight $d(u, v)$.

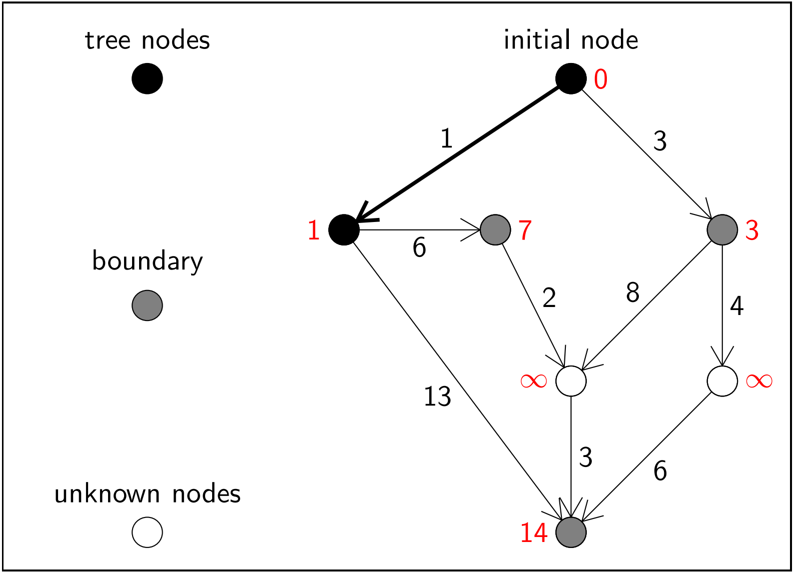
Dijkstra's algorithm

$B := \emptyset$ (tree nodes); $R := \{u\}$ (boundary); $U := V \setminus \{u\}$ (unknown nodes);
$p(u) :=$ **nil**; $D(u) := 0$;
while $R \neq \emptyset$ **do**
  $x :=$ **nil**; $\alpha := \infty$;
  forall $y \in R$ **do**
    if $D(y) < \alpha$ **then**
      $x := y$; $\alpha := D(y)$
    endif
  endfor
  $B := B \cup \{x\}$; $R := R \setminus \{x\}$
  forall $(x, y) \in E$ **do**
    if $y \in U$ **then**
      $D(y) := D(x) + \gamma(x, y)$; $p(y) := x$; $U := U \setminus \{y\}$; $R := R \cup \{y\}$
    elsif $y \in R$ **and** $D(x) + \gamma(x, y) < D(y)$ **then**
      $D(y) := D(x) + \gamma(x, y)$; $p(y) := x$
    endif
  endfor
endwhile

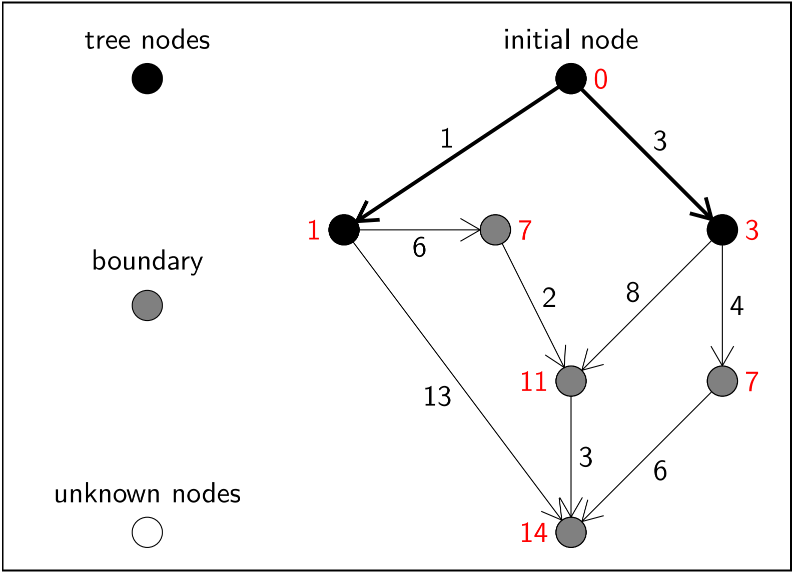
Example for Dijkstra's algorithm



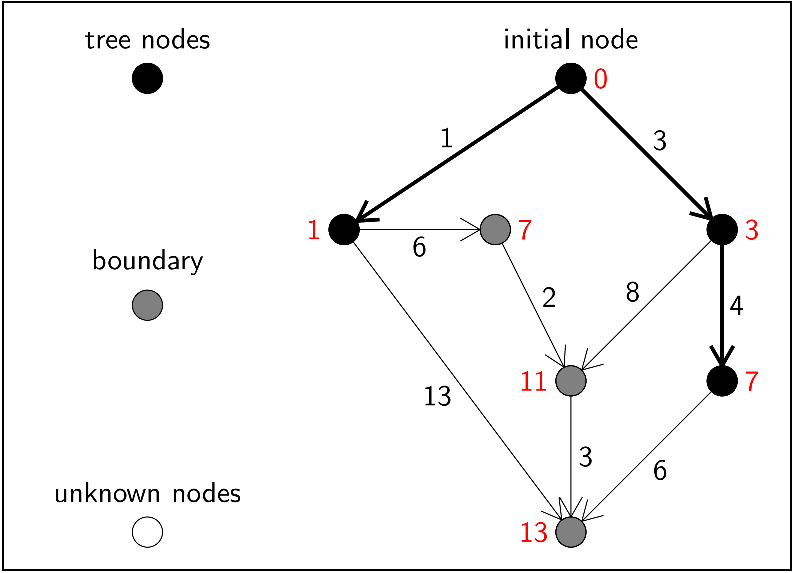
Example for Dijkstra's algorithm



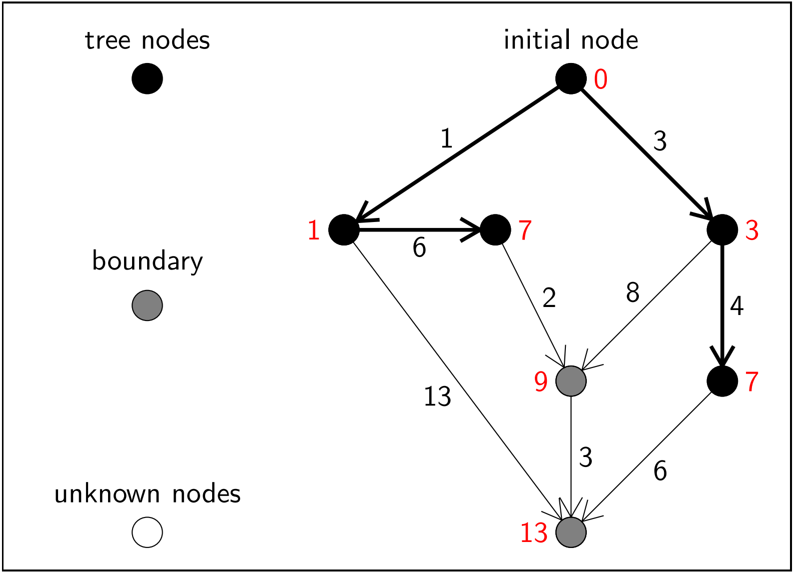
Example for Dijkstra's algorithm



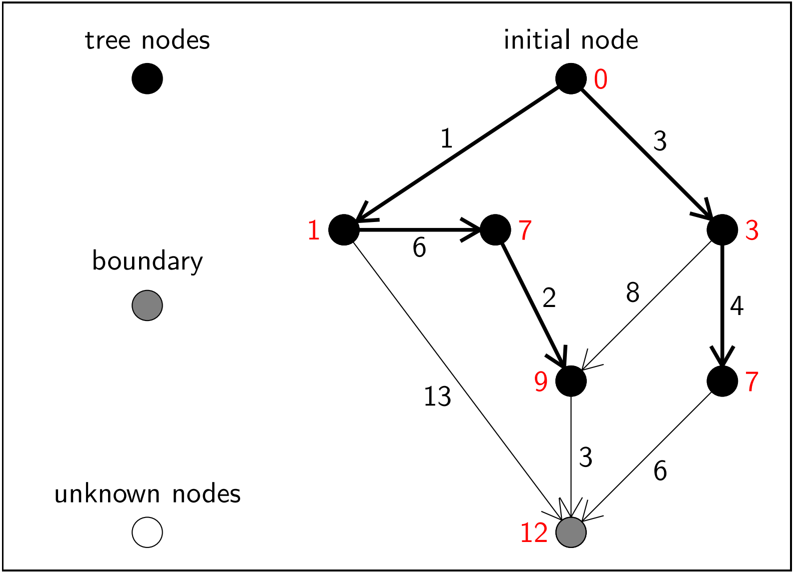
Example for Dijkstra's algorithm



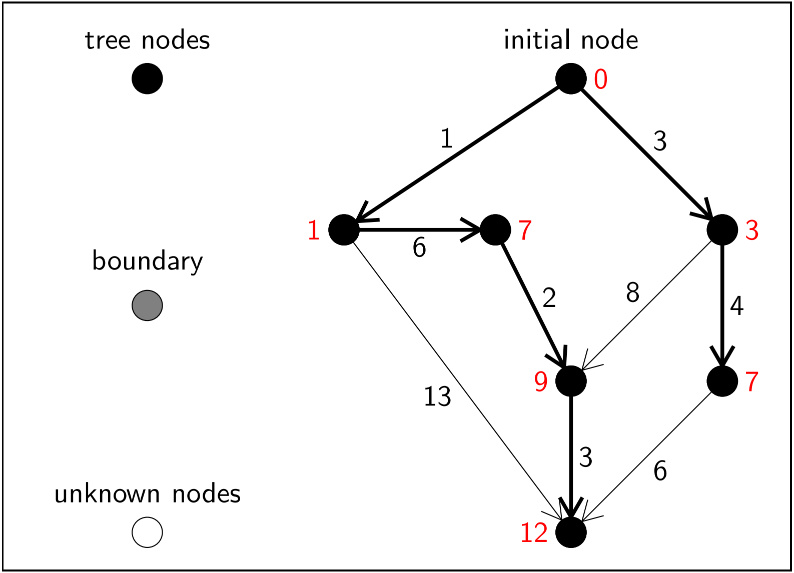
Example for Dijkstra's algorithm



Example for Dijkstra's algorithm



Example for Dijkstra's algorithm



Correctness of Dijkstra's algorithm

Theorem 15 (Correctness of Dijkstra's algorithm)

Dijkstra's algorithm computes shortest paths from the source node to all other nodes.

Proof: We show that the following invariants are preserved by the loop-body of the **while**-loop:

1. The sets B, R, and U form a partition of the node set V.
2. $R = \{y \mid \exists x \in B : (x,y) \in E\} \setminus B$
3. for all $x \in B$, $D(x) = d(u,x)$
4. for all $y \in R$, $D(y) = \min\{D(x) + \gamma(x,y) \mid x \in B, (x,y) \in E\}$

Consider an execution of the body of the **while**-loop, where the node x is moved from R to B.

(1)–(4) hold before the execution of the loop-body.

It is clear that (1) and (2) are preserved.

Correctness of Dijkstra's algorithm

(3): Because of (3) and (4) there exists a node $z \in B$ with

$$D(x) = D(z) + \gamma(z, x) = d(u, z) + \gamma(z, x).$$

Hence, there is path from u to x with weight $D(x)$.

Assume that there is a path from u to x with weight $< D(x)$.

Let $w \in R$ be the first node on this path, which does not belong to B (must exist since $x \notin B$) and let $v \in B$ be the predecessor of w on the path (exists, since $u \in B$).

Since the whole path has weight $< D(x)$, we get

$$\begin{aligned} D(w) &= \min\{D(w') + \gamma(w', w) \mid w' \in B, (w', w) \in E\} \\ &\leq D(v) + \gamma(v, w) < D(x), \end{aligned}$$

which contradicts the choice of $x \in R$.

Hence, we must have $d(u, x) = D(x)$.

Correctness of Dijkstra's algorithm

(4): Let B', R', U', D' be the values of the variables B, R, U, D after the execution of the loop-body.

Note: $B' = B \cup \{x\}$, $D(z) = D'(z)$ for all $z \in B$ and $D(x) = D'(x)$.

Let $y \in R'$.

Case 1: $y \in R \setminus \{x\}$ and $(x, y) \in E$. We have

$$\begin{aligned}
D'(y) &= \min\{D(y), D(x) + \gamma(x, y)\} \\
&= \min\{\min\{D(z) + \gamma(z, y) \mid z \in B, (z, y) \in E\}, D(x) + \gamma(x, y)\} \\
&= \min\{\min\{D'(z) + \gamma(z, y) \mid z \in B, (z, y) \in E\}, D'(x) + \gamma(x, y)\} \\
&= \min\{D'(z) + \gamma(z, y) \mid z \in B', (z, y) \in E\}
\end{aligned}$$

Case 2: $y \in R \setminus \{x\}$ and $(x, y) \notin E$. We have

$$\begin{aligned}
D'(y) &= D(y) \\
&= \min\{D(z) + \gamma(z, y) \mid z \in B, (z, y) \in E\} \\
&= \min\{D'(z) + \gamma(z, y) \mid z \in B', (z, y) \in E\}.
\end{aligned}$$

Correctness of Dijkstra's algorithm

Case 3: $y \notin R$. We have $(x, y) \in E$, but there is no edge $(z, y) \in E$ with $z \in B$ (by (2)).

Hence, we have

$$\begin{aligned} D'(y) &= D(x) + \gamma(x, y) \\ &= D'(x) + \gamma(y, x) \\ &= \min\{D'(z) + \gamma(z, y) \mid z \in B', (z, y) \in E\}. \end{aligned}$$

$\square$

Remark: Dijkstra's algorithm in general does not produce a correct result if negative edge weights are allowed.

Dijkstra with abstract data types for the boundary

In order to analyze the running time of Dijkstra's algorithm, it is useful to reformulate the algorithm with an abstract data type for the boundary R.

The following operations are needed for the boundary R:

insert	insert a new element into R.
decrease-key	decrease the key value of an element of R.
delete-min	find the element from R with the smallest key value and remove it from R.

Dijkstra with abstract data types for the boundary

$B := \emptyset$; $R := \{u\}$; $U := V \setminus \{u\}$; $p(u) :=$ **nil**; $D(u) := 0$;
while $(R \neq \emptyset)$ **do**
  $x :=$ delete-min(R);
  $B := B \cup \{x\}$;
  forall $(x, y) \in E$ **do**
    if $y \in U$ **then**
      $U := U \setminus \{y\}$; $p(y) := x$; $D(y) := D(x) + \gamma(x, y)$;
      insert$(R, y, D(y))$;
    elsif $y \in R$ **and** $D(x) + \gamma(x, y) < D(y)$ **then**
      $p(y) := x$; $D(y) := D(x) + \gamma(x, y)$;
      decrease-key$(R, y, D(y))$;
    endif
  endfor
endwhile

Running time of Dijkstra's algorithm

Number of operations (n = number of nodes, e = number of edges):

insert	n
decrease-key	e
delete-min	n

The total running time depends of the data structure that is used for the boundary:

1. Array of size n:
   single insert/decrease-key: $\mathcal{O}(1)$
   single delete-min: $\mathcal{O}(n)$
   total running time: $\mathcal{O}(n + e + n^2) = \mathcal{O}(n^2)$
2. Heap (balanced binary tree of depth $\mathcal{O}(\log(n))$):
   single insert/decrease-key/delete-min: $\mathcal{O}(\log(n))$
   total running time: $\mathcal{O}(n \log(n) + e \log(n)) = \mathcal{O}(e \log(n))$.

   If $\mathcal{O}(e) \subseteq o(n^2/\log n)$, then the heap beats the array.
   For instance, for planar graphs one has $e \leq 3n - 6$ for $n \geq 3$.

Fibonacci heaps (Fredman & Tarjan 1984)

Fibonacci heaps beat arrays as well as heaps: $\mathcal{O}(e + n \log n)$

A Fibonacci heap H is a list of rooted trees, i.e., a forest.

V is the set of nodes

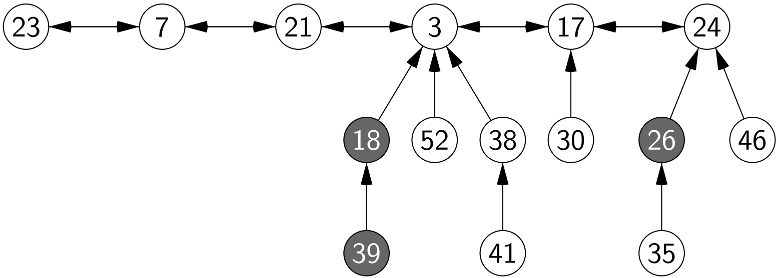
Every node $v \in V$ has a key $key(v) \in \mathbb{N}$.

Heap condition: $\forall x \in V: \; y$ is a child of $x \;\Rightarrow key(x) \leq key(y)$

Some of nodes of V are marked. The root of a tree is never marked.

Example for a Fibonacci heap

(key values are in the circles, marked nodes are grey)



Fibonacci heaps

- The parent-child relation has to be realized by pointers, since the trees in a Fibonacci heap are not necessarily balanced.
- That means that pointer manipulations (expensive!) replace the index manipulations (cheap!) in standard heaps.
- Operations:
  1. **merge**
  2. **insert**
  3. **delete-min**
  4. **decrease-key**

Implementation of **merge** and **insert**

- **merge**: Concatenation of two lists — constant time
- **insert**: Special case of **merge** — constant time
- **merge** and **insert** produce long lists of one-element trees.
- Every such list is a Fibonacci heap.

Implementation of **delete-min**

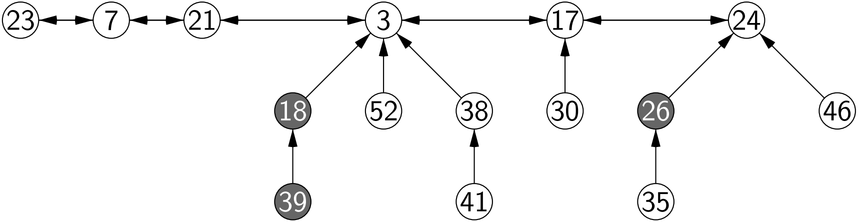
- Let H be a Fibonacci heap consisting of T trees and n nodes.
- for a nodes $x \in V$ let $\text{rank}(x)$ be the number of children of x.
- for a tree B in H let $\text{rank}(B)$ be the rank of the root of B.
- Let $r_{\text{max}}(n)$ be the maximal rank that can appear in a Fibonacci heap with n nodes.
- Clearly, $r_{\text{max}}(n) \leq n$. Later, we will show that $r_{\text{max}}(n) \in \mathcal{O}(\log n)$.

Implementation of **delete-min**

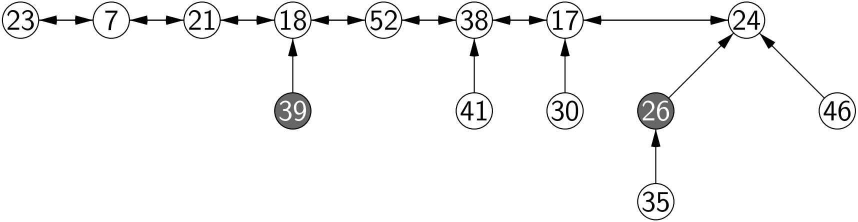
1. Search for the root x with minimal key. Time: $\mathcal{O}(T)$
2. Remove x and replace the subtree rooted in x by its rank(x) many subtrees. Remove possible markings from the new roots.
   Time: $\mathcal{O}(\text{rank}(x)) \subseteq \mathcal{O}(r_{\max}(n))$.
3. Define an array $L[0, \dots, r_{\max}(n)]$, where $L[i]$ is a list of all trees of rank i.
   Time: $\mathcal{O}(T + r_{\max}(n))$.
4. **for** $i := 0$ **to** $r_{\max}(n) - 1$ **do**
   while $|L[i]| \geq 2$ **do**
   remove two trees from $L[i]$
   make the root with the larger key to a child of the other root
   add the resulting tree to $L[i+1]$
   endwhile endfor

   Time: $\mathcal{O}(T + r_{\max}(n))$

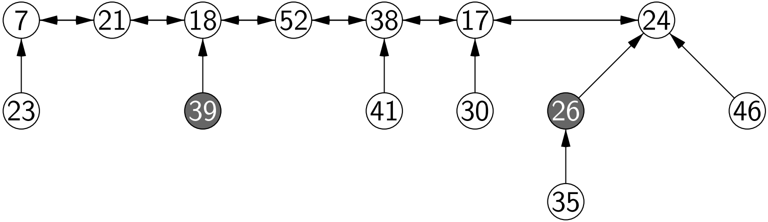
Example for **delete-min**



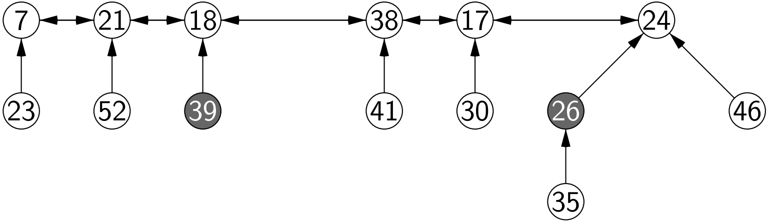
Example for **delete-min**



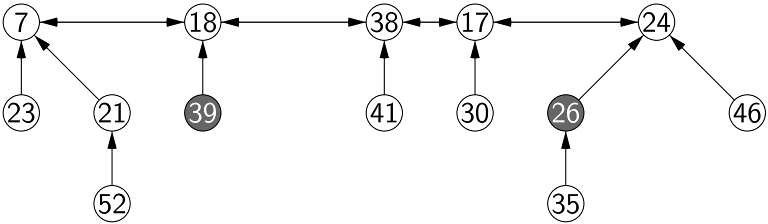
Example for **delete-min**



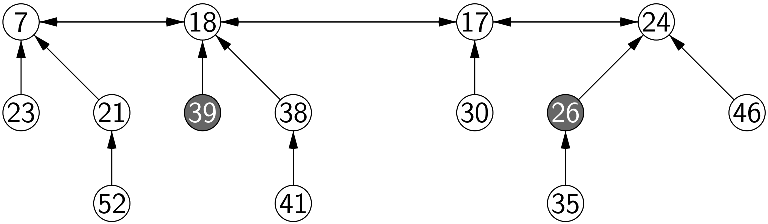
Example for **delete-min**



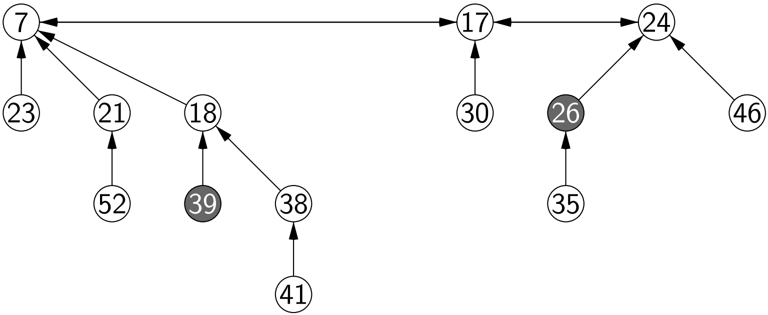
Example for **delete-min**



Example for **delete-min**



Example for **delete-min**



Remarks for **delete-min**

- **delete-min** needs time $\mathcal{O}(T + r_{\max}(n))$, where T is the number of trees before the operation.
- After the execution **delete-min**, there exists for every $i \leq r_{\max}(n)$ at most one tree of rank i.
- Hence, the number of trees after **delete-min** is bounded by $r_{\max}(n)$.

Implementation of **decrease-key**

Let x be the node for which the key is reduced.

1. If x is a root, then we can reduced $key(x)$ without any other modifications.

   Now assume that x is not a root and let $x = y_0, y_1, \ldots, y_m$ be the path from x to the root y_m $(m \geq 1)$.

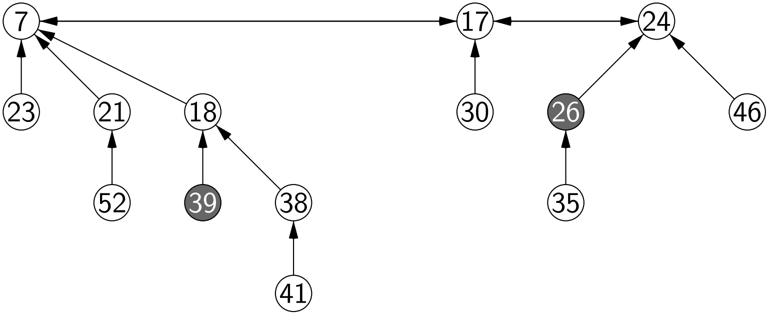
   Let y_k $(1 \leq k \leq m)$ be the first node on this path, which is not x and which is not marked (note: y_m is not marked).

2. For all $0 \leq i < k$, we cut off y_i from its parent node y_{i+1} and remove the marking from y_i ($y_0 = x$ can be marked).

   y_i $(0 \leq i < k)$ is now a unmarked root of a new tree.

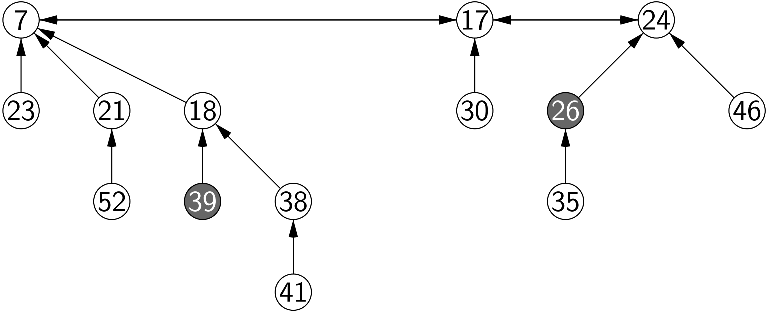
3. If y_k is not a root, then we mark y_k (this tells us later that y_k lost a child).

Example for **decrease-key**



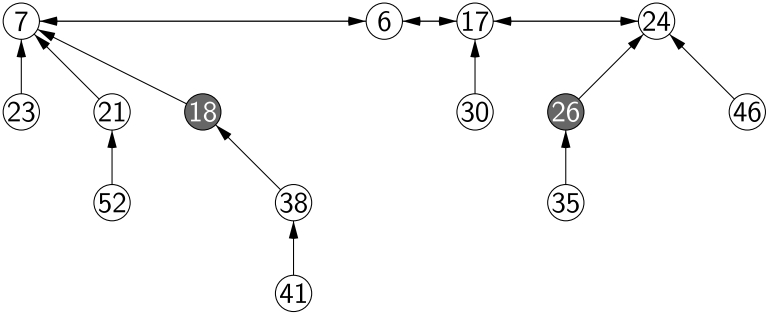
Example for **decrease-key**

decrease-key(node with key 39, 6)



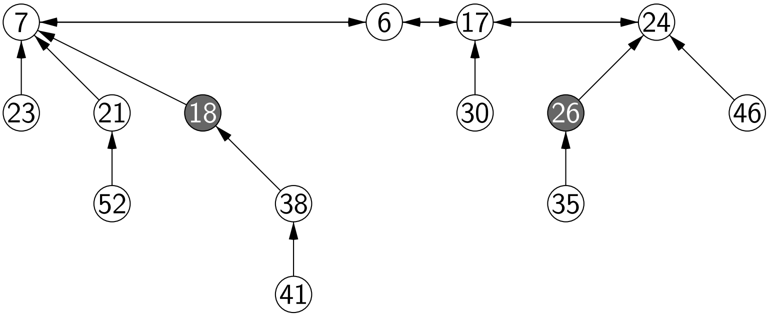
Example for **decrease-key**

decrease-key(node with key 39, 6)



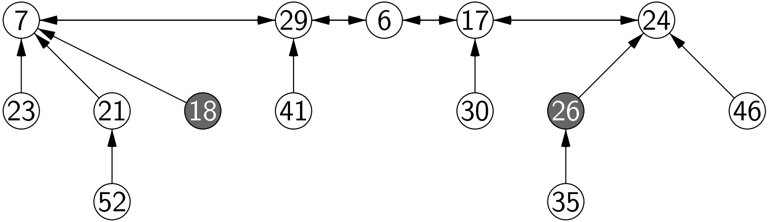
Example for **decrease-key**

decrease-key(node with key 38, 29)



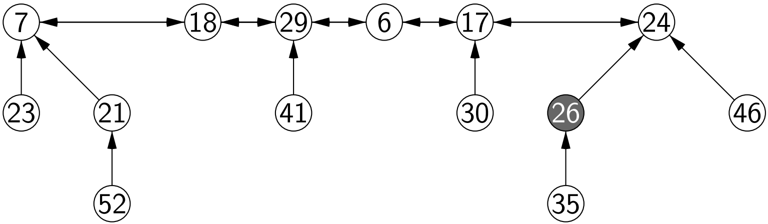
Example for **decrease-key**

decrease-key(node with key $38, 29$)



Example for **decrease-key**

decrease-key(node with key 38, 29)



Remarks for **decrease-key**

- Time: $\mathcal{O}(k) + \mathcal{O}(1)$
- **decrease-key** reduces the number of marked nodes by at least $k - 2$ ($k \geq 1$).
- **decrease-key** increases the number of trees by k.

Definition of Fibonacci heaps

Definition 16

A Fibonacci heap is a list of rooted trees as described before, which can be obtained from the empty list be an arbitrary sequence of **merge**, **insert**, **delete-min**, and **decrease-key** operations

Lemma 17 (Fibonacci heap lemma)

Let x be a node of a Fibonacci heap with $\mathit{rank}(x) = k$.

1. *If $c_1, \ldots, c_k$ are the children of x, and c_i became a child of x before c_{i+1} became a child of x, then $\mathit{rank}(c_i) \geq i - 2$.*
2. *The subtree rooted in x contains at least F_{k+1} many nodes. Here, F_{k+1} is the $(k+1)$-th Fibonacci number ($F_0 = F_1 = 1, F_{k+1} = F_k + F_{k-1}$ for $k \geq 1$).*

Proof of the Fibonacci heap lemma

Part 1:

At the time instant t, where c_i became a child of x, the nodes $c_1, \ldots, c_{i-1}$ were already children of x, i.e., the rank of x at time t was at least $i - 1$.

Since only trees with equal rank are merged to a single tree (in **delete-min**), that rank of c_i at time t was at least $i - 1$ as well.

In the meantime (i.e. after time t), c_i can lose at most one child: If c_i loses one child due to a **decrease-key**, then c_i will be marked, and after loosing second child, c_i will be cut off from the parent node x.

Hence, $\text{rank}(c_i) \geq i - 2$.

Proof of the Fibonacci heap lemma

Part 2:

Proof by induction on the height of the subtree rooted at x.

If x is a leaf, then $k = 0$ and the subtree rooted in x contains $1 = F_1$ node.

If x is not a leaf then we can count the number of nodes in the subtree rooted at x as follows:

1. 2 (for x and c_1) plus
2. the number of nodes in the subtree rooted at c_i (for $2 \leq i \leq k$), which has rank $\geq i - 2$ (by part 1) and therefore contains by induction at least F_{i-1} many nodes.

Hence the subtree rooted in x contains at least

$$2 + \sum_{i=2}^{k} F_{i-1} = 2 + \sum_{i=1}^{k-1} F_i = F_{k+1}$$

many nodes.

Growth of the Fibonacci numbers

Theorem 18

For all $k \geq 0$ we have:

$$F_k = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{k+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{k+1}$$

The Fibonacci numbers grow exponentially.

Consequence: $r_{\max}(n) \in \mathcal{O}(\log n)$.

Summary of the running times

- **merge**, **insert**: constant time
- **delete-min**: $\mathcal{O}(T + r_{\max}(n)) \subseteq \mathcal{O}(T + \log n)$, where T is the current number of trees.
- **decrease-key**: $\mathcal{O}(1) + \mathcal{O}(k)$ $(k \geq 1)$, where at least $k - 2$ markings are removed from the Fibonacci heap.

Amortized time

Definition 19

For a Fibonacci heap H we define its potential $pot(H)$ as $pot(H) := T + 2M$, where T is its number of trees and M is the number of marked nodes.
For an operation op let $\Delta_{pot}(op)$ be the difference of the potential after and before the execution of the operation.

$$\Delta_{pot}(op) = pot(\text{heap after } op) - pot(\text{heap before } op)\,.$$

The amortized time of the operation is op is

$$t_{amort}(op) = t(op) + \Delta_{pot}(op)\,.$$

Amortized time

The potential has the following properties:

- $pot(H) \geq 0$
- $pot(H) \in \mathcal{O}(|H|)$
- $pot(nil) = 0$

Let $op_1, op_2, op_3, \ldots, op_m$ be sequence of m operations, and assume that the initial Fibonacci heap is empty.

We have

$$\sum_{i=1}^{m} t(op_i) \leq \sum_{i=1}^{m} t_{amort}(op_i)\,.$$

Remark: The difference between these two sums is the potential of the generated Fibonacci heap.

Hence, it suffices to bound $t_{amort}(op)$.

Amortized time

Convention: By multiplying all terms in the following computations with a suitable constant, we can assume that

- **merge** and **insert** need one time step,
- that **delete-min** needs $T + \log n$ time steps, and
- that **decrease-key** needs $k + 1$ time steps.

This allows to omit the $\mathcal{O}$-notation.

Amortized time

- $t_{amort}(\textbf{merge}) = t(\textbf{merge}) = 1$, because the potential of the concatenation of two lists is the sum of the potentials of the two lists.
- $t_{amort}(\textbf{insert}) = t(\textbf{insert}) + \Delta_{pot}(op) = 1 + 1 = 2$.
- For **delete-min** we have $t(\textbf{delete-min}) \leq T + \log n$, where T is the number of trees before the execution of **delete-min**.

  After **delete-min** is the number of trees bounded by $r_{\max}(n)$.

  The number of marked nodes can only get smaller.

  Hence, we have $\Delta_{pot}(op) \leq r_{\max}(n) - T$ and $t_{amort}(\textbf{delete-min}) \leq T + \log n - T + r_{\max}(n) \in \mathcal{O}(\log n)$.

Amortized time

- For **decrease-key** we have $t(\textbf{decrease-key}) \leq k+1$ $(k \geq 1)$, where at least $k-2$ markings will be removed.

  Moreover, k new trees are added to the Fibonacci heap.
  We get

$$\begin{aligned} \Delta_{pot}(op) &= \Delta(T) + 2\Delta(M) \\ &\leq k + 2\cdot(2-k) \\ &= 4-k, \end{aligned}$$

  and hence $t_{amort}(\textbf{decrease-key}) \leq k+1+4-k = 5 \in \mathcal{O}(1)$.

Amortized time

Theorem 20

The following amortized time bounds hold for a Fibonacci heap:

$$t_{amort}(\textbf{merge}) \in \mathcal{O}(1)$$
$$t_{amort}(\textbf{insert}) \in \mathcal{O}(1)$$
$$t_{amort}(\textbf{delete-min}) \in \mathcal{O}(\log n)$$
$$t_{amort}(\textbf{decrease-key}) \in \mathcal{O}(1)$$

Fibonacci heaps for Dijkstra

Back to Dijkstra's algorithm:

Let n be the number of nodes and e be the number of edges of the input graph.

Dijkstra's algorithm will execute at most n **insert**-, e **decrease-key**- and n **delete-min**-operations.

$$
\begin{aligned}
t_{\text{Dijkstra}} \quad \leq & \qquad n \cdot t_{amort}(\textbf{insert}) \\
& + \quad e \cdot t_{amort}(\textbf{decrease-key}) \\
& + \quad n \cdot t_{amort}(\textbf{delete-min}) \\
\in & \quad \mathcal{O}(n + e + n \log n) \\
= & \quad \mathcal{O}(e + n \log n)
\end{aligned}
$$

Remember that:

- with arrays we have $t_{\text{Dijkstra}} \in \mathcal{O}(n^2)$, and
- with standard heaps we have $t_{\text{Dijkstra}} \in \mathcal{O}(e \log(n))$.

Idea of dynamic programming

Compute a table of all subsolutions of a problem, until the overall solution is computed.

Every subsolutions is computed using the already existing entries in the table.

Dynamic programming is tightly related to backtracking.

In contrast to backtracking, dynamic programming used iteration instead of recursion. By storing computed subsolutions in table we avoid to solve the same subproblem several times.

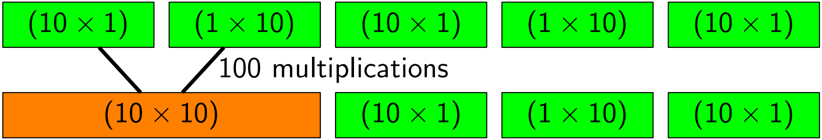
Example: Computing a long product of matrices

Multiplication from left to right:

(10×1) (1×10) (10×1) (1×10) (10×1)

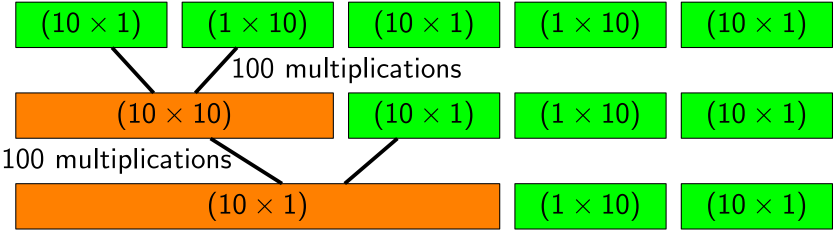
Example: Computing a long product of matrices

Multiplication from left to right:



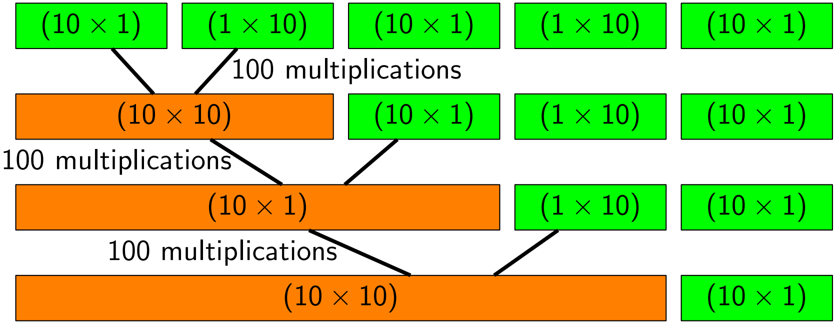
Example: Computing a long product of matrices

Multiplication from left to right:



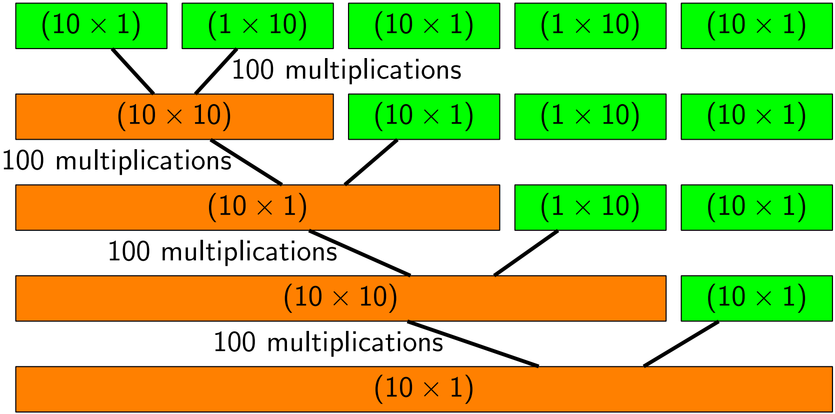
Example: Computing a long product of matrices

Multiplication from left to right:



Example: Computing a long product of matrices

Multiplication from left to right:



In total: **400** multiplications

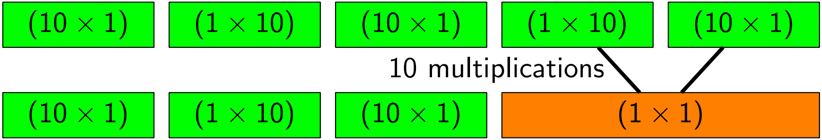
Example: Computing a long product of matrices

Multiplication from right to left:

(10×1)	(1×10)	(10×1)	(1×10)	(10×1)

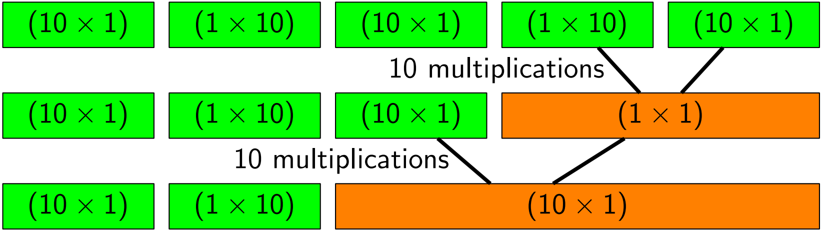
Example: Computing a long product of matrices

Multiplication from right to left:



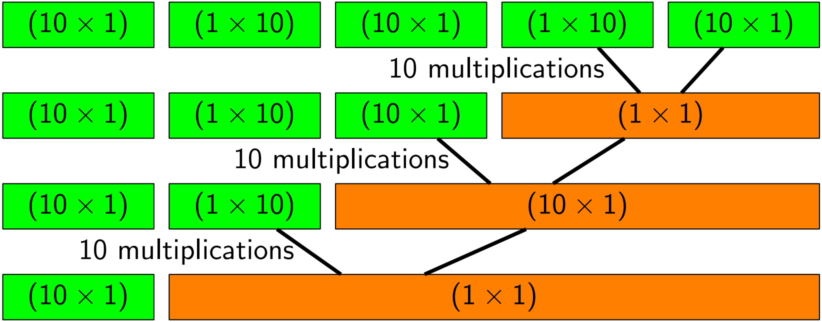
Example: Computing a long product of matrices

Multiplication from right to left:



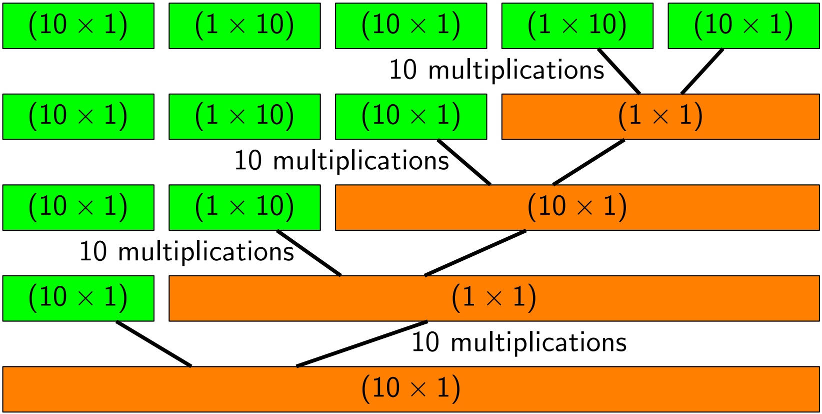
Example: Computing a long product of matrices

Multiplication from right to left:



Example: Computing a long product of matrices

Multiplication from right to left:



In total: **40** multiplications

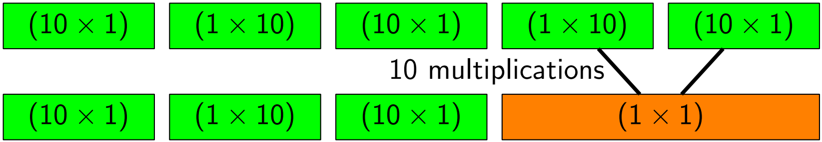
Example: Computing a long product of matrices

Multiplication in **optimal order**

(10×1)	(1×10)	(10×1)	(1×10)	(10×1)

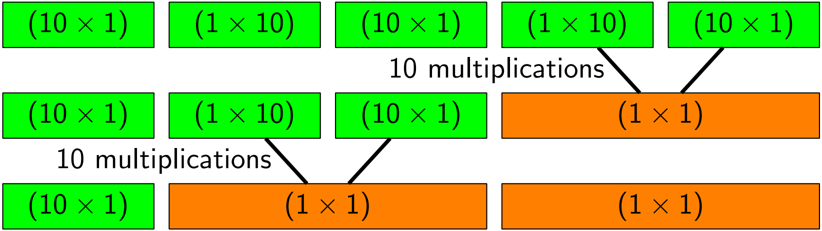
Example: Computing a long product of matrices

Multiplication in **optimal order**



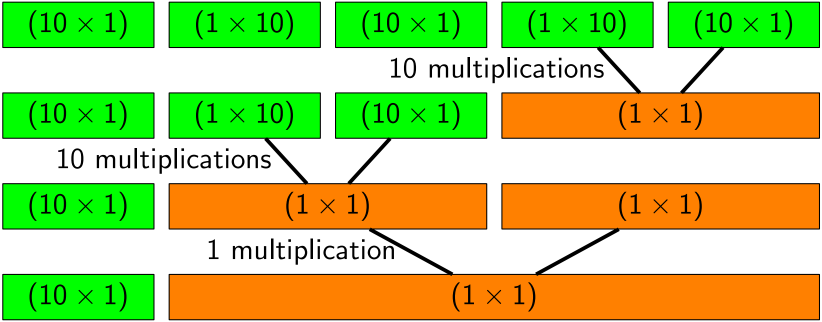
Example: Computing a long product of matrices

Multiplication in **optimal order**



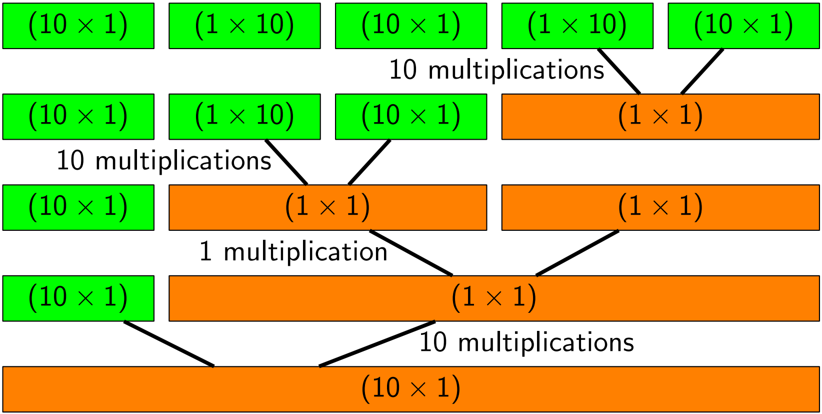
Example: Computing a long product of matrices

Multiplication in **optimal order**



Example: Computing a long product of matrices

Multiplication in **optimal order**



In total: **31** multiplications

Computing a long product of matrices

Let $A_{(n,m)}$ be a matrix with n rows and m columns.

Assumption: Computing $A_{(n,m)} := B_{(n,q)} \cdot C_{(q,m)}$ needs $n \cdot q \cdot m$ scalar multiplications.

Input: sequence of matrices $M^1_{(n_0,n_1)}, M^2_{(n_1,n_2)}, M^3_{(n_2,n_3)}, \ldots, M^N_{(n_{N-1},n_N)}$.

$\mathrm{cost}(M^1, \ldots, M^N) :=$ minimal number of scalar multiplications in order to compute $M^1 \cdots M^N$ (minimum is taken over all possible bracketings).

Dynamic programming approach:

$$\begin{aligned}&\mathrm{cost}(M^i, \ldots, M^j) = \\ &\quad \min_k\{\mathrm{cost}(M^i, \ldots, M^k) + \mathrm{cost}(M^{k+1}, \ldots, M^j) + n_{i-1} \cdot n_k \cdot n_j\}\end{aligned}$$

Computing a long product of matrices

for $i := 1$ **to** N **do**
  $\text{cost}[i, i] := 0$;
  for $j := i + 1$ **to** N **do**
    $\text{cost}[i, j] := \infty$;
  endfor
endfor
for $d := 1$ **to** $N - 1$ **do**
  for $i := 1$ **to** $N - d$ **do**
    $j := i + d$;
    for $k := i$ **to** $j - 1$ **do**
      $t := \text{cost}[i, k] + \text{cost}[k + 1, j] + n[i - 1] \cdot n[k] \cdot n[j]$;
      if $t < \text{cost}[i, j]$ **then**
        $\text{cost}[i, j] := t$;
        $\text{best}[i, j] := k$;
      endif
    endfor
  endfor
endfor
return best

Optimal search trees

We will see a straightforward dynamic programming algorithm for computing optimal search trees with a running time of of $\Theta(n^3)$.

An algorithm of Donald E. Knuth reduces the time to $\Theta(n^2)$ ($\rightarrow$ Algorithms II).

Optimal search trees

Let $V = \{v_1, \ldots, v_n\}$ be linearly ordered set of keys, $v_1 < v_2 < \cdots < v_n$.

For every key $v \in V$ we have given an access probability (also called the weight) $\gamma(v)$.

The idea is that with every key some additional information is associated (think about personnel numbers, and additional informations like name, birthday, salary, etc). Then $\gamma(v_i)$ is the probability that the information associated with key v_i is accessed.

A binary search tree for $v_1 < v_2 < \cdots < v_n$ is a binary tree with node set $\{v_1, v_2, \ldots, v_n\}$, such that:
For every node v with left (resp., right) subtree L (resp. R) and all $u \in L$ (resp. $w \in R$) we have: $u < v$ ($v < w$).

Optimal search trees

Every node v of a search tree B has a level $\ell_B(v)$:
$\ell_B(v) := 1 +$ distance (in number of edges) from v to root.

Finding a node at level ℓ requires ℓ comparisons (start in root and then walk down the path to the node).

Problem: Find a binary search tree B with minimal weighted inner path length

$$P(B) := \sum_{v \in V} \ell_B(v) \cdot \gamma(v).$$

The weighted inner path length is the average cost for accessing a node.

Dynamic programming works because subtrees of optimal binary search trees have to be optimal again.

Optimal search trees

Notation:

- node set $= \{1, \ldots, n\}$, i.e., we identify node v_i with i.
- $P[i,j]$: weighted inner path length of an optimal search tree for the node set $\{i, \ldots, j\}$.
- $R[i,j]$: root of an optimal search tree for $\{i, \ldots, j\}$.

  Since there might be several optimal search trees we take for $R[i,j]$ for the largest root among all optimal search trees.
- $\Gamma[i,j] := \sum_{k=i}^{j} \gamma(k)$: total weight of the node set $\{i, \ldots, j\}$.

Computing optimal search trees in time $\mathcal{O}(n^3)$

Using dynamic programming we can compute all values $P[i,j]$ and $R[i,j]$.

For a binary search tree B with left subtree B_0, right subtree B_1, and total weight $\Gamma(B)$ we have

$$P(B) = P(B_0) + P(B_1) + \Gamma(B).$$

We realize this idea with a cubic algorithm:

- $P[i,j] = \Gamma[i,j] + \min\{P[i,k-1] + P[k+1,j] \mid k \in \{i,\dots,j\}\}$
- $R[i,j] =$ largest key among all k for which $P[i,k-1] + P[k+1,j]$ is minimal.

Optimal search trees

```
for i := 1 to n do
   P[i, i−1] := 0;
   P[i, i] := γ(i);
   Γ[i, i] := γ(i);
   R[i, i] := i;
endfor
```

```
for d := 1 to n−1 do
   for i := 1 to n−d do
      j := i + d;
      root := i;
      t := ∞;
      for k := i to j do
         if P[i, k−1] + P[k+1, j] ≤ t then
            t := P[i, k−1] + P[k+1, j];
            root := k;
         endif
      endfor
      Γ[i, j] := Γ[i, j−1] + γ(j);
      P[i, j] := t + Γ[i, j];
      R[i, j] := root;
   endfor
endfor
```

Computation of regular expressions

Recall from GTI: Computation of regular expressions by Kleene.

A nondeterministic finite automaton (NFA) is a tuple

$$A = (Q, \Sigma, \delta \subseteq Q \times \Sigma \times Q, I, F) \quad (\text{w.l.o.g. } Q = \{1, \dots, n\}).$$

Let $L^k[i,j]$ be the set of all words that label a path in A, which

- leads from i to j and
- thereby only visits intermediate states from $\{1, \dots, k\}$ (i and j do not necessarily belong to $\{1, \dots, k\}$).

Goal: Regular expressions for all $L^n[i,j]$ with $i \in I$ and $j \in F$.

We have

$$\begin{aligned} L^0[i,j] &= \begin{cases} \{a \in \Sigma \mid (i,a,j) \in \delta\} & \text{if } i \neq j \\ \{a \in \Sigma \mid (i,a,j) \in \delta\} \cup \{\varepsilon\} & \text{if } i = j \end{cases} \\ L^k[i,j] &= L^{k-1}[i,j] + L^{k-1}[i,k] \cdot L^{k-1}[k,k]^* \cdot L^{k-1}[k,j] \end{aligned}$$

Computation of regular expressions

Algorithm Regular from an NFA

procedure NFA2REGEXP
Input : NEA $A = (Q, \Sigma, \delta \subseteq Q \times \Sigma \times Q, I, F)$
(Initialize: $L[i,j] := \{a \mid (i,a,j) \in \delta \vee a = \varepsilon \wedge i = j\}$)
begin
  for $k := 1$ **to** n **do**
    for $i := 1$ **to** n **do**
      for $j := 1$ **to** n **do**
        $L[i,j] := L[i,j] + L[i,k] \cdot L[k,k]^* \cdot L[k,j]$
      endfor
    endfor
  endfor
end

Computing the transitive closure

Algorithm Warshall-algorithm: computation of the transitive closure

procedure Warshall (**var** A : adjacency matrix)
Input : graph given by its adjacency matrix $(A[i,j]) \in \mathrm{Bool}_{n \times n}$
begin
  for $k := 1$ **to** n **do**
    for $i := 1$ **to** n **do**
      for $j := 1$ **to** n **do**
        if $(A[i,k] = 1)$ **and** $(A[k,j] = 1)$ **then**
          $A[i,j] := 1$
        endif
      endfor
    endfor
  endfor
end

Transitive closure?

Algorithm Is this algorithm correct?

```
procedure Warshall (var A : adjacency matrix)
Input : graph given by its adjacency matrix (A[i, j]) ∈ Bool_{n×n}
begin
  for i := 1 to n do
    for j := 1 to n do
      for k := 1 to n do
        if (A[i, k] = 1) and (A[k, j] = 1) then
          A[i, j] := 1
        endif
      endfor
    endfor
  endfor
end
```

Correctness of Warshall

Correctness of Warshall's algorithm follows from the following invariant:

1. After the k-th excecution of the body of the **for**-loop, we have: $A[i,j] = 1$, if there is a path from i to j with intermediate nodes from $1, \dots, k$.

   Important: the outermost loop runs over k!

2. If $A[i,j]$ is set to 1, then there exists a path from i to j.

If the 0/1-entries in the adjacency matrix are replaced from edge weights, one obtains Floyd's algorithm for computing shortest paths:

Floyd-algorithm

Algorithm Floyd: all shortest paths in a graph

procedure Floyd (**var** A : adjacency matrix)
Input : edge-weighted graph given by its adjacency matrix $A[i,j] \in (\mathbb{N} \cup \infty)_{n\times n}$, where $A[i,j] = \infty$ means that there is no edge from i to j.
begin
  for $k := 1$ **to** n **do**
    for $i := 1$ **to** n **do**
      for $j := 1$ **to** n **do**
        $A[i,j] := \min\{A[i,j], A[i,k] + A[k,j]\}$;
      endfor
    endfor
  endfor
end

Floyd-algorithm

Floyd's algorithm computes correct results also for graphs with negative weights provide that there do not exist cycles with negative total weight.

Running time of Warshall and Floyd: $\Theta(n^3)$.

„Improvement“ by testing before the j-loop, whether $A[i,k] = 1$ (resp., $A[i,k] < \infty$) holds.

This yields a running time of $\mathcal{O}(n^3)$:

Floyd's algorithm

Algorithm Floyd's algorithm in $\mathcal{O}(n^3)$

procedure Floyd (**var** A : adjacency matrix)
Input : adjacency matrix $A[i,j] \in (\mathbb{N} \cup \infty)_{n \times n}$
begin
  for $k := 1$ **to** n **do**
    for $i := 1$ **to** n **do**
      if $A[i,k] < \infty$ **then**
        for $j := 1$ **to** n **do**
          $A[i,j] := \min\{A[i,j], A[i,k] + A[k,j]\}$;
        endfor
      endif
    endfor
  endfor
end

Floyd's algorithm

Algorithm Floyd's algorithm for negative cycles

procedure Floyd (**var** A : adjacency matrix)
Input : adjacency matrix $A[i,j] \in (\mathbb{Z} \cup \{\infty, -\infty\})_{n\times n}$
begin
  for $k := 1$ **to** n **do**
    for $i := 1$ **to** n **do**
      if $A[i,k] < \infty$ **then**
        for $j := 1$ **to** n **do**
          if $A[k,j] < \infty$ **then**
            if $A[k,k] < 0$ **then** $A[i,j] := -\infty$
              else $A[i,j] := \min\{A[i,j], A[i,k] + A[k,j]\}$
            endif
          endif
  endfor endif endfor endfor
end

Transitive closure and matrix multiplication

Let $A = (a_{i,j})_{1 \leq i,j \leq n}$ be the adjacency matrix of a directed graph with node set $\{1, \ldots, n\}$, i.e.,

$$a_{i,j} = \begin{cases} 1 & \text{if there is an edge from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases}$$

Warshall's algorithm computes the reflexive and transitive closure A^* in time $\mathcal{O}(n^3)$.

Here, $A^* = \sum_{k \geq 0} A^k$, where $A^0 = I_n$ is the identity matrix and $\vee$ (boolean or) is taken for the addition of boolean matrices .

We add as follows: $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1 + 1 = 1$.

By induction we get: $A^k(i,j) = 1 \iff \exists$ path of length k from i to j.

This yields $A^* = \sum_{k=0}^{n-1} A^k$.

Transitive closure and matrix multiplication

Let $B = I_n + A$. We get $A^* = B^m$ for all $m \geq n - 1$.

It therefore suffices to square the matrix $\lceil \log_2(n-1) \rceil$ times in order to compute A^*.

Let $M(n)$ be the time needed to multiply two boolean $n \times n$-matrices and let $T(n)$ be the time needed to compute the reflexive and transitive closure.

We have

$$T(n) \in \mathcal{O}(M(n) \cdot \log n).$$

Using Strassen's algorithm, we get for all $\varepsilon > 0$:

$$T(n) \in \mathcal{O}(n^{\log_2(7)+\varepsilon}).$$

Matrix multiplication $\leq$ transitive closure

Under the plausible assumption that $T(3n) \in \mathcal{O}(T(n))$ we get $M(n) \in \mathcal{O}(T(n))$:

For all boolean matrices A and B we have:

$$\begin{pmatrix} 0 & A & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} I_n & A & AB \\ 0 & I_n & B \\ 0 & 0 & I_n \end{pmatrix}.$$

Under the also plausible assumption that $M(2n) \geq (2+\varepsilon)M(n)$ for an $\varepsilon > 0$, we can show that also $T(n) \in \mathcal{O}(M(n))$.

Hence: The computation of the reflexive and transitive closure is up to constant factors equally expensive as matrix multiplication.

Computation of the transitive closure

Input: $E \in \mathrm{Bool}(n \times n)$

- Divide E into 4 submatrices A, B, C, D such that A and D are square matrices and each of the 4 matrices has size roughly $n/2 \times n/2$:
$$E = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$
- Compute recursively D^*:
Time: $T(n/2)$.
- Compute $F = A + BD^*C$:
Time: $\mathcal{O}(M(n/2)) \leq \mathcal{O}(M(n))$.
- Compute recursively F^*:
Time: $T(n/2)$.
- Set
$$E^* = \left(\begin{array}{c|c} F^* & F^*BD^* \\ \hline D^*CF^* & D^* + D^*CF^*BD^* \end{array}\right).$$

Computation of the transitive closure

We obtain the recurrence

$$T(n) \leq 2T(n/2) + c \cdot M(n) \qquad \text{for some } c > 0.$$

This yields

$$\begin{aligned} T(n) &\leq c \cdot \left(\sum_{i\geq 0} 2^i \cdot M(n/2^i)\right) && \text{(by induction)} \\ &\leq c \cdot \sum_{i \geq 0} \left(\frac{2}{2+\varepsilon}\right)^i \cdot M(n) && \text{(because } M(n/2) \leq \tfrac{1}{2+\varepsilon} M(n) \\ &\in \mathcal{O}(M(n)). \end{aligned}$$

How can we test whether $A \cdot B = C$?

The best current algorithm for multiplying two $(n \times n)$-matrices needs approx. $\Theta(n^{2{,}372873\dots})$ many arithmetic operations (Virginia Vassilevska Williams 2014).

Conjecture

For every $\epsilon > 0$ there exists an algorithm for multiplying two $(n \times n)$-matrices in time $O(n^{2+\varepsilon})$.

Assume now that we have three $(n \times n)$-matrices A, B and C.

How many arithmetic operations are needed to test whether $A \cdot B = C$ holds?

Trivial answer: $O(n^{2{,}372873\dots})$

But there is a better method!

How can we test whether $A \cdot B = C$?

A simple randomized solution:

Algorithm Testing matrix multiplication

---

procedure Test (**var** $A, B, C \in \mathbb{Z}^{n \times n}$: matrices)
begin
  choose $v \in \{0,1\}^{n \times 1}$ uniformly at random
  $w := A \cdot (B \cdot v) - C \cdot v$
  if $w = 0$ **then return** true
  else return false
  endif
end

Computing w requires $O(n^2)$ operations.

How can we test whether $A \cdot B = C$?

Theorem 21

- *If* $A \cdot B = C$ *then the algorithm always returns „true“.*
- *If* $A \cdot B \neq C$ *then the algorithm returns „true“ with probability at most* $1/2$.

The algorithm has a **one-sided error**. By repeating it k times we can reduce the error probability to $\frac{1}{2^k}$.

Proof: Suppose that $D = A \cdot B - C \neq 0$. Let $d \in \mathbb{Z}^{1 \times n}$ be a nonzero-row of D such that $d_k \neq 0$. If $d \cdot v = 0$ for some $v \in \{0,1\}^{n \times 1}$ then $d \cdot v' \neq 0$ where v' is obtained from v by inverting the k-th component. Therefore $\text{Prob}[d \cdot v = 0] \leq 1/2$ and hence $\text{Prob}[w = 0] \leq 1/2$. $\square$

How can we test whether $A \cdot B = C$?

Theorem 22 (Korec, Wiedermann 2014)

Let A, B, C be $(n \times n)$-matrices with entries from $\mathbb{Z}$. Using $O(n^2)$ many operations we can check whether $A \cdot B = C$ holds.

Proof: Let

$$\begin{aligned} A &= (a_{i,j})_{1 \le i,j \le n}, \\ B &= (b_{i,j})_{1 \le i,j \le n}, \\ C &= (c_{i,j})_{1 \le i,j \le n} \text{ and} \\ D &= (d_{i,j})_{1 \le i,j \le n} = A \cdot B - C. \end{aligned}$$

Thus, we have $A \cdot B = C$ if and only if D is the zero-matrix.

Let x be real-valued variable and consider the column-vector

$$v = (1, x, x^2, \dots, x^{n-1})^T.$$

How can we test whether $A \cdot B = C$?

Hence, $D \cdot v = A \cdot B \cdot v - C \cdot v$ is a column-vector whose i-th entry is the polynomial

$$p_i(x) = d_{i,1} + d_{i,2}x + d_{i,3}x^2 + \cdots + d_{i,n}x^{n-1}.$$

We therefore have $A \cdot B = C$ if and only if $p_i(x)$ is the zero polynomial for all $1 \leq i \leq n$.

We use the following theorem:

Cauchy bound

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ be not the zero-polynomial and $a_n \neq 0$. For every α with $p(\alpha) = 0$ we have

$$|\alpha| < 1 + \frac{\max\{|a_i| \mid 0 \leq i \leq n-1\}}{|a_n|}.$$

How can we test whether $A \cdot B = C$?

Proof of the Cauchy bound:

By replacing $p(x)$ by the polynomial $x^n + \frac{a_{n-1}}{a_n} x^{n-1} + \cdots + \frac{a_1}{a_n} x + \frac{a_0}{a_n}$, it suffices to prove the Cauchy bound for the case $a_n = 1$.

Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and

$$h = \max\{|a_i| \mid 0 \le i \le n-1\} > 0.$$

Assume that $p(\alpha) = \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 = 0$, i.e.,

$$\alpha^n = -a_{n-1}\alpha^{n-1} - \cdots - a_1\alpha - a_0.$$

We show that $|\alpha| < 1 + h$.

If $|\alpha| \le 1$, we have $|\alpha| < 1 + h$.

Now assume that $|\alpha| > 1$.

How can we test whether $A \cdot B = C$?

We get

$$\begin{aligned} |\alpha|^n &\leq |a_{n-1}| \cdot |\alpha|^{n-1} + \cdots + |a_1| \cdot |\alpha| + |a_0| \\ &\leq h \cdot (|\alpha|^{n-1} + \cdots + |\alpha| + 1) \\ &= h \cdot \frac{|\alpha|^n - 1}{|\alpha| - 1}. \end{aligned}$$

Since $|\alpha| > 1$, we obtain:

$$|\alpha| - 1 \leq h \cdot \frac{|\alpha|^n - 1}{|\alpha|^n} < h$$

$\square$

How can we test whether $A \cdot B = C$?

Let $a = \max\{|a_{i,j}| \mid 1 \le i,j \le n\}$, $b = \max\{|b_{i,j}| \mid 1 \le i,j \le n\}$ and $c = \max\{|c_{i,j}| \mid 1 \le i,j \le n\}$.

The absolute values of the coefficients of the polynomials $p_i(x)$ can be bounded as follows:

$$|d_{i,j}| = |\sum_{k=1}^{n} a_{i,k} b_{k,j} - c_{i,j}| \le \sum_{k=1}^{n} |a_{i,k}| \cdot |b_{k,j}| + |c_{i,j}| \le n \cdot a \cdot b + c.$$

Let $d = n \cdot a \cdot b + c$ and $r = 1 + d$.

The Cauchy bound yields for all $1 \le i \le n$:

$$p_i(x) = 0 \;\Leftrightarrow\; p_i(r) = 0$$

Note: It is easy to get an upper bound d for the absolute values of the entries of D, but it is not so clear how to get a lower bound.

How can we test whether $A \cdot B = C$?

But: Since A, B, C are matrices over $\mathbb{Z}$, we have $p_i(x) \in \mathbb{Z}[x]$.

Therefore, the absolute value of the leading coefficient of $p_i(x)$ is at least 1, if $p_i(x)$ is not the zero-polynomial.

We can therefore check with the following algorithm, whether $A \cdot B = C$:

1. Compute $a = \max\{|a_{i,j}|\}$, $b = \max\{|b_{i,j}|\}$, $c = \max\{|c_{i,j}|\}$ and $r = 1 + n \cdot a \cdot b + c$
   ($3n^2$ many comparisons, 2 additions, 2 multiplications)
2. Compute the column vector $u = (1, r, r^2, \dots, r^{n-1})^T$.
   ($n - 2$ multiplications)
3. Compute $p := B \cdot u$, $s := A \cdot p$ and $t := C \cdot u$
   ($O(n^2)$ many arithmetic operations)
4. $A \cdot B = C$ if and only if $s = t$.