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Exercise 1

Task 1
Prove or disprove the following statements:

(a) $n \log n \in \mathcal{O}(n^2)$

(b) $n^2 \in \mathcal{O}(n)$

(c) $n^n \in \Omega(2^n)$

(d) $n - \log n \in o(n)$

(e) $2 + (-1)^n \in \Theta(1)$

(f) $n! \in \omega(2^n)$

Solution

(a) True: We know that $\log(n) \leq n$ for all $n \in \mathbb{N}$ with $n \geq 1$, so $n \log(n) \leq n^2$. Therefore, $n \log(n) \in \mathcal{O}(n^2)$ since for $c = 1$ and for all $n \geq n_0 = 1$ it holds that $n \log(n) \leq c \cdot n^2$.

(b) False: Let $n_0 \in \mathbb{N}$, $c > 0$ and assume that for all $n \geq n_0$ it holds that $n^2 \leq c \cdot n$, especially for $n = c + n_0 + 1 \geq n_0$ we have that $(c + n_0 + 1)^2 \leq c \cdot (c + n_0 + 1)$. It follows that $c + n_0 + 1 \leq c$ which is a contradiction.

(c) True: We have $n^n \in \Omega(2^n) \Leftrightarrow 2^n \in \mathcal{O}(n^n)$. We choose $c = 1$ and $n_0 = 2$, then clearly we have $2^n \leq c \cdot 2^n \leq c \cdot n^n$ for all $n \geq n_0$.

(d) False: Let $c = \frac{1}{2}$ and let $n_0 \in \mathbb{N}$. Then $n - \log(n) \leq cn$ means $n \leq 2\log(n)$, which is false for any $n \geq 1$.

(e) True: We have $1 \leq 2 + (-1)^n \leq 3$ for any $n \in \mathbb{N}$. Hence $2 + (-1)^n \leq c \cdot 1$ with $c = 3$, which implies $2 + (-1)^n \in \mathcal{O}(1)$, and $1 \leq c \cdot (2 + (-1)^n)$ with $c = 1$, which implies $1 \in \mathcal{O}(2 + (-1)^n)$.

(f) True: We have $n! \in \omega(2^n) \Leftrightarrow 2^n \in o(n!)$. For $n = 4$ we have $2^n < n!$, since $2^4 = 16 < 24 = 4!$. Therefore $2^n < n!$ for all $n \geq 4$. Hence we also have $2^n < c \cdot n!$ for all $c \geq 1$. Now let $0 < c < 1$ and choose $n_0 = \lceil \frac{1}{c} \rceil \cdot 4$. Clearly $n_0 - 1 \geq 7$. By our observation before we have $2^{n_0 - 1} \leq (n_0 - 1)! \leq c \cdot (n_0 - 1)! \cdot \lceil \frac{1}{c} \rceil$. It follows $2^{n_0} \leq c \cdot n_0!$.

Alternatively you can use lim (or lim sup or lim inf) to prove these statements. For instance

$$\lim_{n \to \infty} \frac{n - \log(n)}{n} = 1 > 0,$$

hence we know $n - \log(n) \notin o(n)$.

Task 2

Let $f\colon \mathbb{N} \to \mathbb{N}$ with $f(n) \in \Theta(n)$. Prove or disprove the following statements:

(a) $f(n)^k \in \Theta(n^k)$ for all $k \in \mathbb{N}$, $k \geq 1$

(b) $2^{f(n)} \in \Theta(2^n)$

Solution

(a) True: From $f(n) \in \Theta(n)$ it follows that:

(1) $f(n) \in \mathcal{O}(n)$, so there exist $n_0 \in \mathbb{N}$, $c > 0$ such that for all $n \geq n_0$ we have $f(n) \leq c \cdot n$, which implies that $f(n)^k \leq (c \cdot n)^k = c^k \cdot n^k$, therefore $f(n)^k \in \mathcal{O}(n^k)$.

(2) $n \in \mathcal{O}(f(n))$, so there exist $n_0 \in \mathbb{N}, c > 0$ such that for all $n \geq n_0$ we have $n \leq c \cdot f(n)$, which implies that $n^k \leq (c \cdot f(n))^k = c^k \cdot f(n)^k$, therefore $n^k \in \mathcal{O}(f(n)^k)$.

We therefore obtain that $f(n)^k \in \Theta(n^k)$.

(b) False: Let $f(n) = 2n$, so $f(n) \in \Theta(n)$. Assume that the statement is true so we have $2^{2n} = 4^n \in \Theta(2^n)$, so $4^n \in \mathcal{O}(2^n)$. This means there exist $n_0 \in \mathbb{N}$, $c > 0$ such that for every $n \geq n_0$ we have $4^n \leq c \cdot 2^n$. This implies $(\frac{4}{2})^n \leq c$. But since 2^n is not a bouned function, this yields a contradiction.

Task 3

Use the Master Theorem to determine the asymptotic growth of the following functions:

(a) $T_1(n) = 7 \cdot T_1\left(\frac{n}{2}\right) + 4n$

(b) $T_2(n) = 7 \cdot T_2\left(\frac{n}{2}\right) + 1000n^2$

(c) $T_3(n) = 8 \cdot T_3\left(\frac{n}{2}\right) + n^2$

(d) $T_4(n) = 8 \cdot T_4\left(\frac{n}{2}\right) + n^3$

(e) $T_5(n) = 6 \cdot T_5\left(\frac{n}{3}\right) + n^3$

(f) $T_6(n) = 64 \cdot T_6\left(\frac{n}{8}\right) + n^2$

Solution

(a) $a = 7$, $b = 2$, $c = 1$. Since $a = 7 > 2 = b^c$, case 3 applies:

$$T_1(n) \in \Theta\left(n^{\frac{\log(a)}{\log(b)}}\right) = \Theta\left(n^{\frac{\log(7)}{\log(2)}}\right).$$

(b) $a = 7, b = 2, c = 2$. Since $a = 7 > 4 = b^c$, case 3 applies:

$$T_2(n) \in \Theta\left(n^{\frac{\log(a)}{\log(b)}}\right) = \Theta\left(n^{\frac{\log(7)}{\log(2)}}\right).$$

(c) $a = 8, b = 2, c = 2$. Since $a = 8 > 4 = b^c$, case 3 applies:

$$T_3(n) \in \Theta\left(n^{\frac{\log(a)}{\log(b)}}\right) = \Theta\left(n^3\right).$$

(d) $a = 8$, $b = 2$, $c = 3$. Since $a = 8 = 2^3 = b^c$, case 2 applies:

$$T_4(n) \in \Theta(n^c \log(n)) = \Theta(n^3 \log(n)).$$

(e) $a = 6$, $b = 3$, $c = 3$. Since $a = 6 < 27 = b^c$, case 1 applies:

$$T_5(n) \in \Theta(n^c) = \Theta(n^3).$$

(f) $a = 64, b = 8, c = 2$. Since $a = 64 = 8^2 = b^c$, case 2 applies:

$$T_6(n) \in \Theta(n^c \log(n)) = \Theta(n^2 \log(n)).$$

Task 4
Sort the array $[2, 8, 13, 4, 7, 16, 3, 12]$ using Mergesort.

Solution
mergesort$(1, 8)$, $m = 4$

- mergesort$(1, 4)$, $m = 2$
  - mergesort$(1, 2)$, $m = 1$, merge$(1, 1, 2)$, $[2, 8]$
  - mergesort$(3, 4)$, $m = 3$, merge$(3, 3, 4)$, $[4, 13]$
  - merge$(1, 2, 4)$, $[2, 4, 8, 13]$
- mergesort$(5, 8)$, $m = 6$
  - mergesort$(5, 6)$, $m = 5$, merge$(5, 5, 6)$, $[7, 16]$
  - mergesort$(7, 8)$, $m = 7$, merge$(7, 7, 8)$, $[3, 12]$
  - merge$(5, 6, 8)$, $[3, 7, 12, 16]$

merge$(1, 4, 8)$, $[2, 3, 4, 7, 8, 13, 16]$