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Exercise 4

Task 1

(a) Show that the leaves of a heap of size n are at positions

$$\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n$$

of the array representation.

(b) How many comparisons does build-heap need on a sorted list?

(c) How many comparisons does build-heap need on a reversed sorted list?

Solution

(a) Let A be the heap in array representation. The children of the node at position i in A are exactly at position $2i$ and $2i+1$ (in a binary tree for the next "layer" you have to multiply the position by 2). Hence i is a leaf in A if and only if $2i > n$.

For $i = \lfloor n/2 \rfloor + 1$ we have $2i \geq n - 1 + 2 = n + 1 > n$. Hence Every node j with $\lfloor n/2 \rfloor + 1 \leq j \leq n$ is a leaf as well.

For $i = \lfloor n/2 \rfloor$ we have $2i \leq n$. Therefore i has at least one child, which means all k with $1 \leq k \leq \lfloor n/2 \rfloor$ cannot be leaves.

(b) At most $2n + 2$ many comparisons (see lecture). For big n this can hardly be beaten, since a sorted list is a min-heap (elements on the top have to sink to the bottom).

(c) At most $2 \cdot \lfloor n/2 \rfloor \leq n$ many comparisons (actually exactly $n - 1$ many!).

Task 2
Sort the following list via Radixsort.

$$[224, 421, 319, 121, 914, 314]$$

Solution

1. - $**1$: 2 elements
   - $**4$: 3 elements
   - $**9$: 1 element

   $\Rightarrow [421, 121, 224, 914, 314, 319]$

2. - $*1*$: 3 elements
   - $*2*$: 3 elements

   $\Rightarrow [914, 314, 319, 421, 121, 224]$

3. - $1**$: 1 element
   - $2**$: 1 element
   - $3**$: 2 elements
   - $4**$: 1 element
   - $9**$: 1 element

   $\Rightarrow [121, 224, 314, 319, 421, 914]$

Task 3
Show that the median of five numbers can be computed using six comparisons.

Solution
The idea is to find two elements of which we know that at least three other elements are greater. Such an element cannot be the median. After this we have three elements remaining. We then have to find out which element among these is the smallest, which is the median.
Let a, b, c, d, e be the five numbers. We assume WLOG that $a < b < c < d < e$ but we have to be careful to discover this through comparisons.

- Compare a and b. Also compare c and d. We now known that $a < b$ and $c < d$.
- Compare the smaller elements of $\{a, b\}$ and $\{c, d\}$ which are a and c. We now known that $a < c$. Since we known that b, c and d are greater than a, it cannot be the median.
- Compare b and e, so we now know that $b < e$.
- Compare the smaller elements of $\{b, e\}$ and $\{c, d\}$ which are b and c. We now know that $b < c$. Since we know that c, d and e are greater than b, it cannot be the median.
- We are now left with $\{c, d, e\}$. The smallest element among these three is the median. We already know that $c < d$, so d cannot be the median. We compare c and e and find out that $c < e$. Therefore, c is the smallest element of $\{c, d, e\}$ and thus is the median.

Task 4
Does the algorithm "Median of the Medians" run in linear time, if one uses blocks of three or blocks of seven?

Solution

- blocks of 3: $T(n) \le T(\frac{n}{3}) + T(\frac{2n}{3}) + c \cdot n$

  The number of comparisons $T(\frac{2n}{3})$ (recursive step) is obtained like on slide 94.

  We cannot use Master Theorem II, so it is not clear, whether $T(n) \in O(n)$ or not.

- blocks of 7: $T(n) \le T(\frac{n}{7}) + T(\frac{5n}{7}) + c \cdot n$

  The number of comparisons $T(\frac{5n}{7})$ (recursive step) is obtained like on slide 94.

  Master Theorem II implies $T(n) \in O(n)$, since $(\frac{1}{7} + \frac{5}{7}) < 1$.

Task 5
Which of the following pairs is a subset system, respectively matroid?

(a) $(\{1, 2, 3\}, \{\emptyset, \{1\}, \{3\}, \{1, 2\}\})$

(b) $(\{1, 2, 3\}, \{\emptyset, \{1\}, \{2\}, \{3\}, \{2, 3\}\})$

(c) (E, U), where E is a finite set and $U = \{A \subseteq E \mid |A| \le k\}$ for a $k \in \mathbb{N}$.

(d) (E, U), where E is a finite set, $\{E_i \mid 1 \le i \le k\}$ is a partition of E and

$$U = \{A \subseteq E \mid |A \cap E_i| \le 1 \text{ for all } 1 \le i \le k\}.$$

Solution
Let E be a finite set and $U \subseteq 2^E$.
A pair (E, U) is a subset system, if $\emptyset \in U$ and $A \subseteq B \in U$ implies $A \in U$.
A subset system (E, U) is a matroid, if for all $A, B \in U$ with $|A| < |B|$ there is an element $x \in B \setminus A$ such that $A \cup \{x\} \in U$.

(a) This is not a subset system, because $\{1, 2\} \in U$ but $\{2\} \notin U$.

(b) This is a subset system. The exchange property for $|\emptyset| < |A|$ for all $A \in U$ is trivial, so we have three cases to check, since $|\{1\}|, |\{2\}|, |\{3\}| < |\{2, 3\}|$. For $\{1\}$ and $\{2, 3\}$ there is no $x \in \{2, 3\} \setminus \{1\} = \{2, 3\}$ such that $\{1\} \cup \{x\} \in U$, since $\{1, 2\} \notin U$ and $\{1, 3\} \notin U$. Therefore, this is not a matroid.

(c) This is a subset system:

   - $\emptyset \in U$ because $\emptyset \subseteq E$ and $|\emptyset| = 0 \le k$.
   - Let $B \in U$, so $B \subseteq E$ and $|B| \le k$. Let $A \subseteq B \subseteq E$. Then $|A| \le |B|$, hence $|A| \le k$ and therefore $A \in U$.

   This is a matroid: Let $A, B \in U$ with $|A| < |B|$. Since $|B| \le k$ we have $|A| < k$, so for every $x \in E$ it holds by definition that $A \cup \{x\} \in U$. Choose any $y \in B \setminus A \neq \emptyset$, hence $A \cup \{y\} \in U$.

(d) This is a subset system:

- $\emptyset \in U$, since $|\emptyset \cap E_i| = 0 \leq 1$ for all $1 \leq i \leq k$.
- Let $B \in U$ and $A \subseteq B$. For all $1 \leq i \leq n$ we have $A \cap E_i \subseteq B \cap E_i$, hence $|A \cap E_i| \leq |B \cap E_i| \leq 1$. Therefore, $A \in U$.

This is a matroid: Let $A, B \subseteq U$ with $|A| < |B|$. Let $I_A \subseteq \mathbb{N}$, $I_B \subseteq \mathbb{N}$ with

$$I_A = \{i \in \{1, \ldots, k\} \mid |A \cap E_i| = 1\}$$
$$I_B = \{i \in \{1, \ldots, k\} \mid |B \cap E_i| = 1\}.$$

Since $|A| < |B|$, there is a $j \in \{1, \ldots, k\}$ with $j \in I_B$ and $j \notin I_A$. Therefore, $|E_j \cap A| = 0$ and $|E_j \cap B| = 1$. This means, we have exactly one element $x \in E_j \cap B$. This already yields $A \cup \{x\} \in U$, since we have $E_j \cap E_i = \emptyset = \{x\} \cap E_i$ for all $1 \leq i \leq k$ with $i \neq j$ ($\{E_i \mid 1 \leq i \leq k\}$ is a partition of E) and $|(A \cup \{x\}) \cap E_j| = 1$.