Exercise 3

Task 1
Perform a division with remainder with Newton's method for $s = 87$ and $t = 7$.

Solution
The goal is to find q and r with $s = qt + r$. We will do $k = \lceil \log_2(\log_2(s)) \rceil = 3$ iterations (note that $\lceil \log_2(\log_2(s)) \rceil \geq \lceil \log(\log(s)) \rceil$). The initial value is $x_0 = \frac{1}{2^3} = \frac{1}{8} \in (\frac{1}{14}, \frac{1}{7}]$ (3 bits). The following 3 terms of the sequence are obtained by the formula $x_{i+1} = 2x_i - 7x_i^2$.

- $x_1 = 2 \cdot \frac{1}{8} - 7 \cdot (\frac{1}{8})^2 = \frac{9}{64}$ (6 bits)
- $x_2 = \frac{585}{4096}$ (12 bits)
- $x_3 = \frac{2396745}{16777216}$ (24 bits)

Therefore we have $s \cdot x_3 \approx 12.43$ and we obtain the value q by either rounding up or down. Testing yields $q = 12$, since $13 \cdot 7 = 91 > 87$. Hence $r = 87 - 12 \cdot 7 = 3$. Finally we have $87 = 12 \cdot 7 + 3$.

Task 2
Let $A \in \mathbb{C}^{n \times n}$ be a matrix.

1. (Slide 61) Show that the coefficient s_1 of the characteristic polynomial $\det(x \cdot \mathrm{Id} - A) = x^n - s_1 x^{n-1} + \cdots$ is equal to the trace of A, which is the sum of the diagonal elements of A.
2. (Lemma 12) Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be the eigenvalues of A ($\lambda_i = \lambda_j$ is allowed). Show that $\lambda_1^m, \ldots, \lambda_n^m$ are the eigenvalues of A^m for $m \in \mathbb{N}$.

Solution

1. By the Leibniz formula we see that we have to evaluate only one term of the determinant $\det(x \cdot \mathrm{Id} - A)$. This is because we need the coefficient of x^{n-1} and all x are only on the main diagonal of $(x \cdot \mathrm{Id} - A)$. But picking any nontrivial permutation $\sigma \in S_n$ means that there are at most $n-2$ many terms in the product

$$\prod_{i=1}^{n} (\delta_{i,\sigma(i)} x - a_{i,\sigma(i)}) \quad (\delta_{i,j} = 1 \Leftrightarrow i = j)$$

containing x. This means the monomial in the procuct with the highest degree is x^j with $j \leq n-2$.

Hence we have to evaluate $\prod_{i=1}^{n}(x - a_{i,i})$ in order to find the coefficient s_1. But indeed, now it becomes clear that

$$\prod_{i=1}^{n}(x - a_{i,i}) = x^n + (-a_{1,1} - \cdots - a_{n,n})x^{n-1} + \cdots = x^n - \mathrm{tr}(A)x^{n-1} + \cdots .$$

The sign of id is 1 and thus we can conclude $s_1 = \mathrm{tr}(A)$.

2. Let λ be an eigenvalue of A. This means there exists a nonzero vector v, such that $Av = \lambda v$. We show that λ^m is an eigenvalue of A^m via a simple induction on m. The case $m = 1$ is trivial. Furthermore we get $A^m v = A^{m-1}(Av) = A^{m-1}(\lambda v) = \lambda A^{m-1} v$. By the induction hypothesis we have $A^{m-1}v = \lambda^{m-1}v$ and hence the assumption follows. We just proved the general case, hence we also proved Lemma 12, since we can replace λ by any λ_i.

Task 3

Invert the following matrix A using Csansky's algorithm.

$$A = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$$

Solution

1. Compute the powers A^1 and
$$A^2 = \begin{pmatrix} 8 & 6 \\ 6 & 5 \end{pmatrix}.$$

2. Compute the traces of these two matrices:
$$\mathrm{tr}(A) = 2 + 1 = 3 \qquad \mathrm{tr}(A^2) = 8 + 5 = 13$$

3. Compute the s_k, $k \in \{1, 2\}$. We have $s_1 = \mathrm{tr}(A) = 3$ and
$$s_2 = \frac{1}{2}(s_1 \mathrm{tr}(A) - \mathrm{tr}(A^2)) = -2.$$
Note that $s_2 = \det(A)$.

4. We obtain A^{-1} by the formula
$$A^{-1} = \frac{-1}{-2}(A - 3 \cdot \mathrm{Id}) = \begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & -1 \end{pmatrix}.$$