Universität Siegen
Lehrstuhl Theoretische Informatik
Markus Lohrey

Exercise 4

Task 1
Consider the following algorithm, which tests probabilistically if $AB = C$ for given matrices $A, B, C \in \mathbb{Z}^{n\times n}$:

1. Choose a vector $v \in \{0,1\}^{n\times 1}$ randomly and uniformly distributed.
2. Compute $w = A(Bv) - Cv$.
3. If $w = 0$ return "yes", otherwise "no".

Prove that in the case $AB \neq C$ the algorithm returns "yes" with a probability of at most $\frac{1}{2}$.

Solution
Let $D = AB - C \neq 0$ and let $d \in \mathbb{Z}^{1\times n}$ be a nonzero row of D with $d_k \neq 0$. Note that w can also be obtained by computing $w = Dv$ (using the distributive law), but this is exactly what we want to avoid in order to have a faster algorithm. This means $d \cdot v$ yields one entry of w, which cannot be 0 for all choices of v.
Let $v \in \{0,1\}^{n\times 1}$. We consider the probability of $dv = 0$. Let $v' = e_k - v$ be the vector, which is obtained by a bit flip of the k-th entry of v. For every v with $dv = 0$ we get $dv' \neq 0$. Hence, at most half of the vectors $v \in \{0,1\}^{n\times 1}$ satisfy $dv = 0$.

Task 2
Let $G = (V, E)$ be an undirected graph with

$$V = \{1,2,3,4,5,6\}, \quad E = \{\{1,3\},\{1,6\},\{2,3\},\{2,5\},\{3,5\},\{4,6\},\{5,6\}\}.$$

(a) Compute the Tutte matrix T_G of G.

(b) Compute the polynomial $\det(T_G)$.

(c) Does G have a perfect matching? If yes, name all perfect matchings of G. If no, justify your answer.

Solution

(a) The Tutte matrix of G is

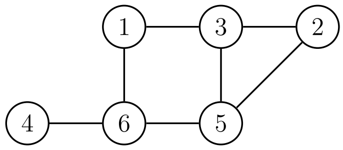
$$T_G = \begin{pmatrix} 0 & 0 & x_{1,3} & 0 & 0 & x_{1,6} \\ 0 & 0 & x_{2,3} & 0 & x_{2,5} & 0 \\ -x_{1,3} & -x_{2,3} & 0 & 0 & x_{3,5} & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{4.6} \\ 0 & -x_{2,5} & -x_{3,5} & 0 & 0 & x_{5.6} \\ -x_{1,6} & 0 & 0 & -x_{4.6} & -x_{5.6} & 0 \end{pmatrix}$$

(b) We use the Laplace expansion (in each step by rows) in order to obtain

$$\begin{aligned}\det(T_G) &= x_{4.6} \cdot x_{1,3} \cdot ((-1) \cdot x_{2,5}) \cdot (-x_{2,5}) \cdot (-x_{1,3}) \cdot (-x_{4,6}) \\ &= x_{1,3}^2 x_{2,5}^2 x_{4,6}^2 \neq 0\end{aligned}$$

(c) Yes, G has a perfect matching. By looking at $\det(T_G)$ it also becomes clear that there is exactly one perfect matching M, namely $M = \{\{1,3\},\{2,5\},\{4,6\}\}$.

For better visualization, this is how G looks like:



Task 3

Let $G = (V, E)$ be an undirected graph with $V = \{1, \dots, n\}$. Let $T^G = (T_{u,v})_{1 \leq u,v \leq n}$ be the matrix defined by

$$T_{u,v} = \begin{cases} x_{u,v} & \text{if } \{u,v\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Let G be a bipartite graph. This means, there are disjoint subsets $U, W \subset V$ such that $V = U \cup W$ and $\{u, w\} \in E$ only if $u \in U$ and $w \in W$ (or $w \in U$ and $u \in W$). Show that G has a perfect matching if and only if $\det(T^G) \neq 0$.

(b) Does (a) also hold, if G is not bipartite?

Solution

(a) In the polynomial $\det(T^G)$ each monomial consists of different combinations of variables. Hence it is impossible that two monomials can cancel out. If M is a perfect matching, then $\det(T^G)$ contains the monomial

$$\prod_{\substack{(u,v) \in V \times V \\ \{u,v\} \in M}} x_{u,v}.$$

This can easily be seen by considering that for every $x_{u,v}$ in the product we will also find the variable $x_{v,u}$ and hence every $v \in V$ can be found at index position 1 and 2 (row and column in the matrix T^G). Note that this is true for any graph G and any perfect matching M of G.

Now let $\det(T^G) \neq 0$. Since G is bipartite, we have a partition $V = U \cup W$ and T^G has a very special form. Namely every $T_{u,v} = 0$ with $u, v \in U$ or $u, v \in W$. W.l.o.g.

$U = \{1, 2, \dots, i\}$ and $W = \{i+1, \dots, n\}$. This means, T^G looks like this:

$$\begin{pmatrix} 0 & \cdots & 0 & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & * & \cdots & * \\ * & \cdots & * & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ * & \cdots & * & 0 & \cdots & 0 \end{pmatrix}$$

If $i > \frac{n}{2}$ (or $i < \frac{n}{2}$), then we definitely find linearly dependent vectors in the first i columns (in the last $n-i$ columns). Hence the determinant is 0 in these cases, which is a contradiction to our assumption. Whence $\det(T^G) \neq 0$ implies $i = \frac{n}{2}$. Using the Leibniz formula, we see that from a monomial of the determinant we obtain a $\sigma \in S_n$, where $\sigma : U \to W$ is a bijection. This yields a perfect matching $M_\sigma = \{\{j, \sigma(j)\} | j \leq \frac{n}{2}\}$.

(b) No: Consider the graph K_3 (which is a triangle). Then $\det(T^{K_3}) = x_{1,2}x_{2,3}x_{3,1} + x_{1,3}x_{2,1}x_{3,2} \neq 0$, but this graph does not have a perfect matching.