Test Exam for „Algorithms I"

WS 2021/22 / February 4, 2022

First name: ____________________

Second name: ____________________

Matriculation numberr: ____________________

task	**max. points**	**points achieved**
1	6	
2	6	
3	6	
4	6	
5	8	
6	6	
Σ	38	

Important information

- Duration of the exam: **60 minutes**. The exam take places on February 24, 2022, (today) from 12 noon to 1 pm German time (duration: 1 hour) in the form of a take home exam.
- Your solutions must be hand written. Please write clearly! You do not have to print out the exam questions. You can write your solution on your own sheets of paper.
- Please write on **each sheet of paper** your **name** and your **matriculation number**. Every solution must be clearly marked with the corresponding **number of the exam question**.
- You have to scan or take photos of your solutions. We prefer to receive a single PDF file and we recommend to use the free Adobe Scan App. Alternatively, you can also write your solutions directly with a tablet and send us a PDF.
- Your solutions must arrive no later than **by 1:20 pm** German time on February 24, 2022, (today) at one of the following email addresses:

  (a)

  (b)

  (c)

  (d)

- Together with your solutions you have to send us the filled and signed declaration of independent performance for the exam.
- You can use all kinds of support (technical tools, lecture notes, books etc.) for the exam **except for the help of other persons**.
- In case of technical problems we will be available by phone via one of the following numbers:

  (a)

  (b)

  (c)

  Note that we cannot guarantee to be always available.

Task 1. (6 Points)
Which of the following statements are correct?

1. $n + \log(n) \in o(n)$
2. $n \in \mathcal{O}(n + \log(n + 1))$
3. $n^n \in o(2^{n \log n})$
4. $\cos(n) + 2 \in \Theta(\sin(n) + 2)$

You do not have to prove your answers.

Solution:

1. wrong (the quotient $\frac{n+\log n}{n} = 1 + \frac{1}{\log n}$ does not converge to zero)
2. correct ($n \leq n + \log(n + 1)$ for all $n \geq 0$)
3. wrong ($2^{n \log n} = n^n$ and $f(n) \in o(f(n))$ is always wrong)
4. correct ($\cos(n) + 2 \leq 3(\sin(n) + 2)$ and $\sin(n) + 2 \leq 3(\cos(n) + 2)$)

Task 2. (6 Points)

Use the Master Theorem to determine the asymptotic growth of the following functions:

1. $T_1(n) = 4 \cdot T_1(n/2) + n^3$
2. $T_2(n) = 16 \cdot T_2(n/4) + n^2$
3. $T_3(n) = 28 \cdot T_3(n/3) + n^3$
4. $T_4(n) = 200 \cdot T_4(n/100) + n$

You do not have to compute the values of logarithms (e.g., just write $\log_2 3$ instead of $1,58496\cdots$).

Solution:

1. $4 < 2^3$, hence $T_1(n) = \Theta(n^3)$
2. $16 = 4^2$, hence $T_2(n) = \Theta(n^2 \log n)$
3. $28 > 3^3$, hence $T_3(n) = \Theta(n^{\log_3 28})$
4. $200 > 100^1$, hence $T_4(n) = \Theta(n^{\log_{100} 200})$

Task 3. (6 Points)

Consider the array A with the following entries:

	1	2	3	4	5	6	7	8	9	10
A	3	2	7	6	5	10	9	8	4	1

Use the Build Heap procedure in order to transform it into a heap. Show the array after each call of reheap.

Solution: We have $n = 10$.

- After reheap$(5, 10)$:

	1	2	3	4	5	6	7	8	9	10
A	3	2	7	6	5	10	9	8	4	1

- After reheap$(4, 10)$:

	1	2	3	4	5	6	7	8	9	10
A	3	2	7	8	5	10	9	6	4	1

- After reheap$(3, 10)$:

	1	2	3	4	5	6	7	8	9	10
A	3	2	10	8	5	7	9	6	4	1

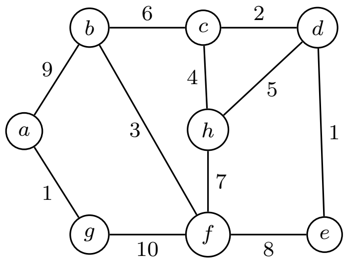
- After reheap$(2, 10)$:

	1	2	3	4	5	6	7	8	9	10
A	3	8	10	6	5	7	9	2	4	1

- After reheap$(1, 10)$:

	1	2	3	4	5	6	7	8	9	10
A	10	8	9	6	5	7	3	2	4	1

Task 4. (6 Points)
Compute a spanning subtree of minimal weight using Kruskal's algorithm for the following graph.



Show the edge selected in each step or indicate if no edge is selected in a step.

Solution: Let T be set current set of selected edges. Initially we have $S = \emptyset$

1. Select edge $\{a, g\}$ (alternatively we good select $\{d, e\}$).
$$T = \{\{a, g\}\}.$$
2. Select edge $\{d, e\}$.
$$T = \{\{a, g\}, \{d, e\}\}.$$
3. Select edge $\{c, d\}$.
$$T = \{\{a, g\}, \{d, e\}, \{c, d\}\}.$$
4. Select edge $\{b, f\}$.
$$T = \{\{a, g\}, \{d, e\}, \{c, d\}, \{b, f\}\}.$$
5. Select edge $\{c, h\}$.
$$T = \{\{a, g\}, \{d, e\}, \{c, d\}, \{b, f\}, \{c, h\}\}.$$
6. Edge $\{d, h\}$ is not selected.
7. Select edge $\{b, c\}$.
$$T = \{\{a, g\}, \{d, e\}, \{c, d\}, \{b, f\}, \{c, h\}, \{b, c\}\}.$$
8. Edge $\{h, f\}$ is not selected.
9. Edge $\{e, f\}$ is not selected.
10. Select edge $\{a, b\}$.
$$T = \{\{a, g\}, \{d, e\}, \{c, d\}, \{b, f\}, \{c, h\}, \{b, c\}, \{a, b\}\}.$$

Task 5. (8 Points)

Assume the following 4 integer matrices are given:

- (10×2)-matrix A_1
- (2×3)-matrix A_2
- (3×8)-matrix A_3
- (8×10)-matrix A_4

In which way would you compute the product $A_1A_2A_3A_4$ so that the total number of integer multiplications is minimized?

Solution:

- (10×3)-matrix A_1A_2: $10 \cdot 2 \cdot 3 = 60$ multiplications
- (2×8)-matrix A_2A_3: $2 \cdot 3 \cdot 8 = 48$ multiplications
- (3×10)-matrix A_3A_4: $3 \cdot 8 \cdot 10 = 240$ multiplications

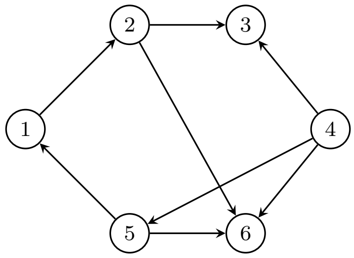
- $(A_1A_2)A_3$: $60 + 10 \cdot 3 \cdot 8 = 300$ multiplications
- $A_1(A_2A_3)$: $48 + 10 \cdot 2 \cdot 8 = 208$ multiplications
- Therefore: best way to compute the (10×8)-matrix $A_1A_2A_3$ is $A_1(A_2A_3)$ and this needs 208 multiplications

- $(A_2A_3)A_4$: $48 + 2 \cdot 8 \cdot 10 = 208$ multiplications
- $A_2(A_3A_4)$: $240 + 2 \cdot 3 \cdot 10 = 300$ multiplications
- Therefore: best way to compute the (2×10)-matrix $A_2A_3A_4$ is $(A_2A_3)A_4$ and this needs 208 multiplications

- $A_1(A_2A_3A_4)$: $208 + 10 \cdot 2 \cdot 10 = 408$ multiplications
- $(A_1A_2)(A_3A_4)$: $60 + 240 + 10 \cdot 3 \cdot 10 = 600$ multiplications
- $(A_1A_2A_3)A_4$: $208 + 10 \cdot 8 \cdot 10 = 1008$ multiplications
- Hence, the best way to compute the (10×10)-matrix $A_1A_2A_3A_4$ is $A_1((A_2A_3)A_4)$ and this needs 408 multiplications.

Task 6. (6 Points)

Let G be the following directed graph:



- Determine the corresponding adjacency matrix of G.
- Compute the adjacency matrix of G^+ (the transitive closure of G).

Solution: The adjacency matrix of G is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The adjacency matrix of G^+ is

$$A^+ = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$